

QUASIHOMOGENEITY OF ISOLATED HYPERSURFACE SINGULARITIES AND LOGARITHMIC COHOMOLOGY

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ABSTRACT. We characterize quasihomogeneity of isolated hypersurface singularities by the injectivity of the map induced by the first differential of the logarithmic differential complex in the top local cohomology supported in the singular point.

1. FORMULATION OF THE RESULT

We consider the germ of an isolated hypersurface singularity $D \subseteq (\mathbb{C}^n, 0)$. Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n, 0}$ be the ring of germs of holomorphic functions on \mathbb{C}^n at the origin with maximal ideal \mathfrak{m} and $f \in \mathfrak{m}^2$ a reduced defining equation of D . For notational simplicity, we shall write \mathbb{C}^n instead of $(\mathbb{C}^n, 0)$. In terms of coordinates x_1, \dots, x_n on \mathbb{C}^n , $\mathcal{O} = \mathbb{C}\{x_1, \dots, x_n\}$ is the ring of convergent power series and $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$. We abbreviate by $f_i = \frac{\partial f}{\partial x_i}$, $i = 1, \dots, n$, the partial derivatives of f .

Let $\Omega^\bullet = \Omega_{\mathbb{C}^n, 0}^\bullet$ be the complex of holomorphic differential forms at the origin and denote by $\Omega^\bullet(\star D) = \Omega^\bullet[f^{-1}]$ the complex of meromorphic differential forms with poles along D . K. Saito [Sai80, §1] introduced the subcomplex of logarithmic differential forms $\Omega^\bullet(\log D) \subseteq \Omega^\bullet(\star D)$ along D . It is the maximal subcomplex consisting of forms with at most a simple pole along D . Let $\text{Der} = \text{Der}_{\mathbb{C}} \mathcal{O}$ be the module of \mathbb{C} -linear derivations of \mathcal{O} . It is a free \mathcal{O} -module of rank n with a basis formed by the partial derivatives $\partial_1, \dots, \partial_n$ with respect to x_1, \dots, x_n . The module of logarithmic vector fields $\text{Der}(-\log D) \subseteq \text{Der}$ consists of all vector fields tangent to the smooth part of D . Under the natural inner product, $\text{Der}(-\log D)$ and $\Omega^1(\log D)$ are mutually \mathcal{O} -dual.

Our object of interest in this article is the natural map

$$H_0^n(\mathcal{O}) \xrightarrow{d_1} H_0^n(\Omega^1(\log D))$$

induced by the first differential of $\Omega^\bullet(\log D)$ in the top local cohomology supported at the origin. We shall show that its injectivity is equivalent to the quasihomogeneity of the singularity.

Theorem 1. *D is quasihomogeneous if and only if d_1 is injective.*

For non-isolated singularities, at least one implication fails: The divisor $D \subseteq \mathbb{C}^3$ defined by $f = xy(x+y)(x+yz)$ is not quasihomogeneous but d_1 is injective.

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2. RELATION TO THE LOGARITHMIC COMPARISON THEOREM

The map d_1 is well-known in the context of the logarithmic comparison theorem: By Grothendieck's comparison theorem [Gro66], the cohomology of the complement $U = \mathbb{C}^n \setminus D$ of a germ of a reduced divisor $D \subseteq \mathbb{C}^n$ is computed by the complex $\Omega^\bullet(\star D)$. If the inclusion $\Omega^\bullet(\log D) \subseteq \Omega^\bullet(\star D)$ is a quasi-isomorphism then $\Omega^\bullet(\log D)$ computes the cohomology of U and one says that the logarithmic comparison theorem, or briefly LCT, holds for D . The characterization of LCT is an open problem and the subject of active current research [Tor04]. Mainly the two extremal cases of isolated singularities and free divisors have been studied. By definition, D is free if $\text{Der}(-\log D)$ is \mathcal{O} -free of rank necessarily equal to n . Freeness of D means that it is either smooth or its singular locus $\text{Sing}(D)$ is of pure codimension 1 in D . Thus the two cases coincide for singularities in dimension $n = 2$ and are disjoint if $n \geq 3$.

By [CJNMM96, CMMNMCJ02, Sect. 2], the map d_1 occurs as the differential $d_1^{0,n-1}$ in the 1-page

$$E_1^{p,q} = H^q(\mathbb{C}^n \setminus \{0\}, \Omega^p(\log D))$$

of a spectral sequence E which converges to the cohomology of U under the assumption that LCT holds for $D \setminus \{0\} \subseteq \mathbb{C}^n \setminus \{0\}$. In particular, this assumption is fulfilled in the case of isolated singularities. Since sections of $\Omega^\bullet(\log D)$ defined outside $\text{Sing}(D)$ extend uniquely to \mathbb{C}^n , the $q = 0$ row of E_1 equals $\Omega^\bullet(\log D)$. Thus, if LCT holds also at the origin, the spectral sequence \bar{E} with 1-page \bar{E}_1 obtained from E_1 by removing the $q = 0$ row converges to zero.

In the case of a free divisor D , $\Omega^\bullet(\log D)$ consists of free \mathcal{O} -modules and hence $E_1^{p,q} = 0$ for $q \neq 0, n-1$. This implies that the injectivity of d_1 is a necessary condition for LCT [CMMNMCJ02, Sect. 3]. In dimension $n = 2$, injectivity of d_1 is even equivalent to LCT and to quasihomogeneity [CMMNMCJ02, Thm. 3.5].

If $f \in \mathfrak{m}_x J_f$ at any point $x \in D$ for a reduced defining equation $f \in \mathfrak{m}$ of D then D is called strongly Euler homogeneous. In the case of free divisors, the main conjecture on LCT states that LCT implies strong Euler homogeneity [CMMNMCJ02, Conj. 1.4]. We have proved this conjecture in dimension $n = 3$ [GS06].

For isolated singularities, strong Euler homogeneity reduces to quasihomogeneity by [Sai71]. M. Holland and D. Mond [HM98] have given an explicit characterization of LCT for quasihomogeneous isolated singularities D : Let $f \in \mathfrak{m}^2$ be a reduced defining equation of D which is quasihomogeneous with respect to the necessarily strictly positive weights $w_1, \dots, w_n \in \mathbb{Q}$. Then LCT holds for D if and only if

$$(\mathcal{O}/\langle f_1, \dots, f_n \rangle)_{k - \sum_{i=1}^n w_i} = 0 \text{ for } k = 1, \dots, n-2.$$

In particular, quasihomogeneity does not imply LCT for isolated singularities. On the other hand, we do not know a counter-example to the opposite implication. Our result possibly reduces the complete characterization of LCT for isolated singularities to the statement: LCT implies injectivity of d_1 for isolated singularities. But it is not clear if the injectivity of d_1 contradicts to \bar{E} converging to zero.

3. PROOF OF THE THEOREM

We denote the n th local cohomology module of \mathcal{O} supported at the origin by

$$H := H_0^n(\mathcal{O}) = H^{n-1}(\mathbb{C}^n \setminus \{0\}, \mathcal{O}) = \frac{\mathcal{O}[\frac{1}{x_1 \cdots x_n}]}{\sum_{i=1}^n \mathcal{O}[\frac{1}{x_1 \cdots \hat{x}_i \cdots x_n}]}.$$

By right exactness of the tensor product $\otimes := \otimes_{\mathcal{O}}$, this implies

$$H_0^n(\Omega^1(\log D)) = H^{n-1}(\mathbb{C}^n \setminus \{0\}, \Omega^1(\log D)) = \Omega^1(\log D) \otimes H.$$

The total differential $d : \mathcal{O} \rightarrow \Omega^1(\log D)$ induces a natural map

$$d_1 : H = H_0^n(\mathcal{O}) \rightarrow H_0^n(\Omega^1(\log D)) = \Omega^1(\log D) \otimes H$$

in local cohomology. The perfect pairing $\text{Der}(-\log D) \otimes \Omega^1(\log D) \rightarrow \mathcal{O}$ [Sai80, Lem. 1.6] identifies $\Omega^1(\log D)$ to $\text{Hom}(\text{Der}(-\log D), \mathcal{O})$. Applying $\otimes H$, it induces a pairing

$$(1) \quad \langle, \rangle : \text{Der}(-\log D) \otimes H_0^n(\Omega^1(\log D)) = \text{Der}(-\log D) \otimes (\Omega^1(\log D) \otimes H) \rightarrow H.$$

such that $\langle \delta, (\omega \otimes [g]) \rangle = \langle \delta, \omega \rangle [g]$. In particular, $d_1([g]) = \sum_{i=1}^n dx_i \otimes [\partial_i(g)]$ for $g \in \mathcal{O}[\frac{1}{x_1 \cdots x_n}]$ and hence

$$(2) \quad \langle \delta, d_1([g]) \rangle = \sum_{i=1}^n \langle \delta, dx_i \rangle [\partial_i(g)] = \delta([g]).$$

Denoting $\text{Hom} = \text{Hom}_{\mathcal{O}}$, the pairing (1) corresponds to the natural map

$$(3) \quad \text{Hom}(\text{Der}(-\log D), \mathcal{O}) \otimes H = \Omega^1(\log D) \otimes H \xrightarrow{\pi} \text{Hom}(\text{Der}(-\log D), H).$$

According to [HM98, Lem. 1.11], $\Omega^1(\log D) = \mathcal{O} \cdot \frac{df}{f} + \Omega^1$ and there is an exact sequence

$$(4) \quad 0 \longrightarrow \mathcal{O} \xrightarrow{\alpha^t} \mathcal{O}^{n+1} \cong \mathcal{O} \oplus \Omega^1 \xrightarrow{\beta} \Omega^1(\log D) \longrightarrow 0$$

where $\alpha = (f, f_1, \dots, f_n)$ and $\beta = (\frac{df}{f}, -dx_1, \dots, -dx_n)$. Dualizing against \mathcal{O} yields

$$0 \longrightarrow \text{Der}(-\log D) \xrightarrow{\beta^t} \mathcal{O}^{n+1} \xrightarrow{\alpha} J \longrightarrow 0, \quad J := \langle f, f_1, \dots, f_n \rangle,$$

and $\text{Ext}^1(\Omega^1(\log D), \mathcal{O}) \cong \mathcal{O}/J$. Dualizing against \mathcal{O} and H yields the sequences

$$(5) \quad 0 \longrightarrow \text{Hom}(J, \mathcal{O}) \longrightarrow \mathcal{O}^{n+1} \xrightarrow{(\beta^t)^t} \text{Hom}(\text{Der}(-\log D), \mathcal{O}) \longrightarrow 0,$$

$$(6) \quad 0 \longrightarrow \text{Hom}(J, H) \longrightarrow H^{n+1} \longrightarrow \text{Hom}(\text{Der}(-\log D), H) \longrightarrow \text{Ext}^1(J, H) \longrightarrow 0.$$

By reflexivity of $\text{Der}(-\log D)$ [Sai80, Cor. 1.7], the sequences (4) and (5) coincide. In particular, $(\beta^t)^t = \beta$ is surjective and $\text{Hom}(J, \mathcal{O}) = \mathcal{O}$ as the kernel of β which is directly obvious. From sequences (5) and (6), we obtain the commutative diagram

$$(7) \quad \begin{array}{ccccc} \text{Hom}(J, \mathcal{O}) \otimes H & \longrightarrow & H^{n+1} & \xrightarrow{\beta \otimes 1} & \text{Hom}(\text{Der}(-\log D), \mathcal{O}) \otimes H \\ \downarrow & & \parallel & & \downarrow \pi \\ \text{Hom}(J, H) & \xrightarrow{\iota} & H^{n+1} & \xrightarrow{\gamma} & \text{Hom}(\text{Der}(-\log D), H) \end{array}$$

involving the map π in (3).

Lemma 2. *If D is not Euler homogeneous then $\text{tr } \delta_0 = 0$ for all $\delta \in \text{Der}(-\log D)$. In particular, $\pi(d_1([\frac{1}{x_1 \cdots x_n}]))(\delta) = \langle \delta, d_1([\frac{1}{x_1 \cdots x_n}]) \rangle = 0$ for all $\delta \in \text{Der}(-\log D)$.*

Proof. By the formal structure theorem [GS06, Thm. 5.4], there is a system of generators \mathcal{S} of the \mathfrak{m} -adic completion $\widehat{\text{Der}}(-\log D)$ of $\text{Der}(-\log D)$ such that, for $\delta \in \mathcal{S}$, either δ_0 is nilpotent and hence $\text{tr } \delta_0 = 0$ or $\delta = \delta_0$ is semisimple and $\delta(\hat{f}) \in \mathbb{Q} \cdot \hat{f}$ where $\hat{f} := \hat{u} \cdot f$ for some unit $\hat{u} \in \widehat{\mathcal{O}}^*$. In the second case, $\delta \in \text{Ann}_{\widehat{\text{Der}}}(\hat{f})$ if D and hence \hat{f} is not Euler homogeneous. Since D is an isolated singularity,

$$\text{Ann}_{\widehat{\text{Der}}}(\hat{f}) = \langle \hat{f}_i \partial_j - \hat{f}_j \partial_i \mid 1 \leq i < j \leq n \rangle, \quad \hat{f}_i := \partial_i(\hat{f}).$$

By a formal coordinate change [Arn72, Lem. 4.1], $\hat{f} = x_1^2 + \dots + x_k^2 + g(x_{k+1}, \dots, x_n)$ with $g \in \widehat{\mathfrak{m}}^3$ and $\text{tr } \delta_0 = 0$ follows also in the second case from

$$\delta_0 \in \text{Ann}_{\widehat{\text{Der}}}(\hat{f})_0 = \langle x_i \partial_j - x_j \partial_i \mid 1 \leq i < j \leq k \rangle.$$

Since $\text{Der}(-\log D)_0 = \widehat{\text{Der}}(-\log D)_0$ and by additivity and coordinate invariance of $\text{tr } \delta_0$, this proves the first claim. The second follows from (2) and the obvious formula $\delta([\frac{1}{x_1 \dots x_n}]) = \text{tr}(\delta_0)[\frac{1}{x_1 \dots x_n}]$ for any $\delta \in \mathfrak{m} \cdot \text{Der}$. \square

We continue our study of diagram (7) under the assumption that D is not Euler homogeneous. By Lemma 2, $\pi(d_1([\frac{1}{x_1 \dots x_n}])) = 0$ and $d_1([\frac{1}{x_1 \dots x_n}]) = (\beta \otimes 1)(\psi)$ where $\psi = (0, u_1, \dots, u_n) \in H^{n+1}$ and $u_i = -\partial_i([\frac{1}{x_1 \dots x_n}]) = [\frac{1}{x_1 \dots x_i^2 \dots x_n}]$. Then $\gamma(\psi) = 0$ and hence $\psi = \iota(\varphi)$ where $\varphi \in \text{Hom}(J, H)$ is characterised by

$$(8) \quad \varphi(f_i) = \left[\frac{1}{x_1 \dots x_i^2 \dots x_n} \right], \quad \varphi(f) = 0.$$

By injectivity of ι , $d_1([\frac{1}{x_1 \dots x_n}]) = 0$ if and only if φ extends to \mathcal{O} .

Lemma 3. *The map $\varphi \in \text{Hom}(J, H)$ which fulfils (8) extends to a $\tilde{\varphi} \in \text{Hom}(\mathcal{O}, H)$.*

Proof. Denote by $A := \mathbb{C}[x_1, \dots, x_n] \subseteq \mathcal{O}$ the polynomial ring and by $M^* := \text{Hom}_{\mathbb{C}}(M, \mathbb{C})$ the algebraic dual of an A - or \mathcal{O} -module M . The residue pairing [Ems78, Sect. B.1] $\langle \cdot, \cdot \rangle : H \otimes_{\mathbb{C}} \mathcal{O} \rightarrow \mathbb{C}$ is defined by $\langle [g], [h] \rangle := \text{res}([g \cdot h])$ where the residue map $\text{res} : H \rightarrow \mathbb{C}$ sends $[g] \in H$ to the coefficient of $\frac{1}{x_1 \dots x_n}$ in g . More explicitly,

$$\left\langle \left[\sum_{i \in \mathbb{N}^n} \frac{a_i}{x_1^{i_1+1} \dots x_n^{i_n+1}} \right], \sum_{i \in \mathbb{N}^n} b_i x_1^{i_1} \dots x_n^{i_n} \right\rangle = \sum_{i \in \mathbb{N}^n} a_i b_i.$$

The residue pairing defines an injection of H into \mathcal{O}^* . Restricted to A , the pairing injects H into $A^* = \mathbb{C}[[x_1^{-1}, \dots, x_n^{-1}]] \cdot \frac{1}{x_1 \dots x_n}$. By (8), φ annihilates the ideal $K := \langle f \rangle + \mathfrak{m}^2 \cdot \langle f_1, \dots, f_n \rangle$ and induces a $\tilde{\varphi} \in \text{Hom}(J/K, H)$. Since K has finite colength, $\mathfrak{m}^{N+1} \subseteq K$ for some $N > 0$ and hence the finite vector space

$$K^\perp \subseteq (\mathfrak{m}^{N+1})^\perp = \bigoplus_{|i| \leq N} \frac{\mathbb{C}}{x_1^{i_1+1} \dots x_n^{i_n+1}}$$

identifies to $(K \cap A)^\perp \subseteq A^*$. Then the residue pairing induces an isomorphism

$$\ell : K^\perp = (K \cap A)^\perp \rightarrow (A/K \cap A)^* = (\mathcal{O}/K)^*$$

defined by $\ell([g])([h]) = \langle [g], [h] \rangle = \text{res}([g \cdot h])$ for $[g] \in K^\perp$ and $[h] \in \mathcal{O}/K$. Thus, the linear form $\text{res} \circ \tilde{\varphi} \in (J/K)^*$ extends to $\ell([g])$ for some $[g] \in K^\perp$ which means that $\text{res}(\varphi(h)) = \text{res}([g \cdot h])$ for all $h \in J$. Now, define $\tilde{\varphi} \in \text{Hom}(\mathcal{O}, H)$ by $\tilde{\varphi}(h) := [g \cdot h]$ for all $h \in \mathcal{O}$. Then, for $h \in J$,

$$\langle \tilde{\varphi}(h), x^\alpha \rangle = \langle [g \cdot h], x^\alpha \rangle = \text{res}([g \cdot x^\alpha h]) = \text{res}(\varphi(x^\alpha h)) = \langle \varphi(h), x^\alpha \rangle$$

for all $\alpha \in \mathbb{N}^n$ and hence $\tilde{\varphi}(h) = \varphi(h)$ which implies that $\tilde{\varphi}$ is an extension of φ . \square

Proof of Theorem 1. For isolated singularities, Euler and quasihomogeneity are equivalent by [Sai71] and $h^p(H_0^n(\Omega^\bullet(\log D))) = 0$ for all p by [HM98, Cor. 1.8]. For non Euler homogeneous D , $0 \neq [\frac{1}{x_1 \cdots x_n}] \in \ker d_1$ by Lemma 3. \square

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