Gröbner basics for mixed Hodge modules

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Abstract

We develop a Gröbner basis theory for a class of algebras that generalizes both PBW-algebras and rings of differential algebras on smooth varieties. Emphasis lies on methods to compute filtrations and graded structures defined by weight vectors. The approach is tailored for bifiltered $\mathcal{D}$-modules satisfying properties of mixed Hodge modules. As a key ingredient in functors of such modules our theory applies to compute the order filtration on pieces of a V-filtration.

Keywords: Gröbner basis, D-module, Hodge module, PBW-algebra
2010 MSC: 13P10, 14F10, 16W70, 32S35

Introduction

Most algorithms in algebraic $\mathcal{D}$-module theory are based on translating $\mathcal{D}$-module theoretic constructions to computationally accessible operations over the ring of differential operators on the affine n-space which agrees with the Weyl algebra (see e.g. Oaku and Takayama (2001)). However, this approach limits the underlying varieties often to affine n-spaces. To deal with such constructions for general smooth varieties X, we work on a covering of X by affine open sets U on which the tangent sheaf is $\mathcal{O}_U$-free and glue the results. Each such U can be seen as a closed set in an affine n-space and we lift a basis of the tangent sheaf to elements $y_1, \ldots, y_m$ of the Weyl algebra. Then $\mathcal{D}_X(U)$ becomes the factor algebra of the free associative $\mathbb{C}$-algebra $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$ by the two-sided ideal generated by the defining ideal $I(U) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ of U and the natural commutation relations of the variables. We refer to such an algebra as a coordinate system algebra. As opposed to Weyl algebras, coordinate system

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algebras are in general not quotients of PBW-algebras. While there is a Gröbner basis theory for PBW-algebras, we are not aware of a well-developed generalization covering coordinate system rings. We remark that algorithms by Oaku (1996) for coordinate system algebras are not sufficiently general for our purpose (see Remark 1.18).

In this article we develop a Gröbner basis theory for so-called PBW-reduction-algebras which form a common generalization of PBW-algebras and coordinate system algebras (see Section 1). Combined with gluing techniques this allows for $\mathcal{D}$-module calculations on general smooth varieties (Rottner, 2018). Such an algebra is a certain quotient of a free associative $\mathbb{K}$-algebra of type $\mathbb{K}\langle x_1, \dots, x_n\rangle$ by a two-sided ideal containing commutation relations with the property that a subset of the set of standard monomials $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha \in \mathbb{N}^n\}$ forms a $\mathbb{K}$-basis. Our Gröbner basis methods rely on so-called PBW-reduction data. While their existence is guaranteed, we know only in special cases how to compute them. We describe a method in case of quotients of PBW-algebras and coordinate system algebras. Generalizing the underlying notions of standard representation and s-polynomial, we give a Buchberger criterion for PBW-reduction-algebras for well-orderings. As a consequence we obtain Gröbner basics for PBW-reduction-algebras.

Our methods are designed for application to mixed Hodge modules as defined by Saito (1990). These objects can be considered as bifiltered $\mathcal{D}_X$-modules $\mathcal{M}$ with an order and a V-filtration satisfying certain compatibilities. A technical ingredient in the construction of Hodge theoretic functors such as the direct image functor and the vanishing cycle functor is the induced order filtration on the $V_0\,\mathcal{D}_X$-submodule $V_\alpha\,\mathcal{M}$ for $\alpha \in \mathbb{Q}$. Our goal to compute this filtration gives the direction for the following sections. On a suitable coordinate system algebra D the two filtrations are induced by weight vectors $\mathbf{w}$ and $\mathbf{v}$ on the algebra generators $x_1, \dots, x_n$ and $\mathcal{M}(U)$ can be represented as a quotient D^E/Q with an induced order filtration.

In more generality, we consider a PBW-reduction-algebra A with two filtrations $F^{\mathbf{v}}_\bullet A$ and $F^{\mathbf{w}}_\bullet A$ given by weight vectors $\mathbf{v}$ and $\mathbf{w}$. Motivated by our problem in Hodge theory we impose among other conditions the inclusions of algebras $F^{\mathbf{w}}_0 A \subseteq F^{\mathbf{v}}_0 A \subseteq A$. Given a finitely presented A-module M with induced $F^{\mathbf{w}}_\bullet A$-filtration $F^{\mathbf{w}}_\bullet M$ and an $F^{\mathbf{v}}_0 A$-submodule V of M, our main algorithm (see Algorithm 4.12) yields the $F^{\mathbf{w}}_\bullet F^{\mathbf{v}}_0 A$-submodule filtration $F^{\mathbf{w}}_\bullet V$ on V if it is a good filtration. It computes for increasing integers k the intersection $F^{\mathbf{w}}_k V$ of the $F^{\mathbf{v}}_0 A$-submodule V and the $F^{\mathbf{w}}_0 A$-submodule $F^{\mathbf{w}}_k M$ (see Section 4). A stopping criterion (Corollary 4.10) serves to check whether $F^{\mathbf{w}}_k V$ already generates $F^{\mathbf{w}}_\bullet V$. Using a common bound for the $\mathbf{v}$-degree of V and $F^{\mathbf{w}}_k M$, the computation of their intersection is reduced to the case where M is a free A-module (see Section 3). Here we reformulate it as an intersection problem over the subalgebra $F^{\mathbf{v}}_0 A$ of A. We show that $F^{\mathbf{v}}_0 A$ is again a PBW-reduction-algebra and require $F^{\mathbf{w}}_\bullet F^{\mathbf{v}}_0 A$ to be a weight filtration on it. We may thus assume that $F^{\mathbf{v}}_0 A = A$. Finally, we want to intersect an A-submodule and an $F^{\mathbf{w}}_0 A$-submodule of a free A-module. This is achieved by a syzygy computation combined with a Gröbner basis computation with respect to a $\mathbf{w}$-degree ordering. In general this might not be a well-ordering. To address this issue, we describe homogenization techniques for PBW-reduction-algebras (see Section 2).

The results presented in this article originate from the Ph.D. thesis of the first named author (Rottner, 2018) which was supervised by the second named author.

1. Gröbner basis framework for PBW-reduction-algebras

1.1. PBW-reduction-algebras

We first fix some notation: By E we usually denote a finite set of module generators on which we pick a total order $<^E$. Given a ring R and a left R-module M, we write M^E for the direct sum

$\bigoplus_{e\in E} M(e)$, where (e) stands for the free generator corresponding to $e \in E$. We identify the R-modules R and R^E in case E has cardinality one. For a subset $S \subseteq M$ we define similarly $S^E := \{s(e) \mid s \in S, e \in E\} \subseteq M^E$. A map of R-modules $\psi : M \to N$ induces naturally a map $\psi^E : M^E \to N^E$. Given finite sets $E_1, \ldots, E_s$ and $1 \le i_1 < \cdots < i_l \le s$, we denote by $\pi_{E_{i_1},\ldots,E_{i_l}}$ the projection of $R^{E_1} \oplus \cdots \oplus R^{E_s}$ to $R^{E_{i_1}} \oplus \cdots \oplus R^{E_{i_l}}$. Given a left, right and two-sided R-module M and a subset $N \subseteq M$, we denote by ${}_R\langle N\rangle$, $\langle N\rangle_R$ and ${}_R\langle N\rangle_R = \langle N\rangle$ the left, right and two-sided R-submodule of M generated by N, respectively. Unless said otherwise, we mean by an R-module always a left R-module.

Throughout we denote by $\mathbb{K}$ a field and by $x_1, \ldots, x_n$ variables. We consider the polynomial ring $\mathbb{K}[\underline{x}] := \mathbb{K}[x_1, \ldots, x_n]$ as $\mathbb{K}$-submodule of the free associative $\mathbb{K}$-algebra $\mathbb{K}\langle\underline{\mathrm{x}}\rangle := \mathbb{K}\langle x_1, \ldots, x_n\rangle$.

Definition 1.1. Let E be a finite set.

1. We denote by

$$\mathrm{Mon}(\mathbb{K}[\underline{x}]^E) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n}(e) \mid \alpha \in \mathbb{N}^n, e \in E\} \subseteq \mathbb{K}[\underline{x}]^E$$

and

$$\mathrm{Mon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle^E) := \{x_{i_1} \cdots x_{i_k}(e) \mid k \in \mathbb{N}, 1 \le i_1, \ldots, i_k \le n, e \in E\} \subseteq \mathbb{K}\langle\underline{\mathrm{x}}\rangle^E$$

the set of *monomials* of $\mathbb{K}[\underline{x}]^E$ and $\mathbb{K}\langle\underline{\mathrm{x}}\rangle^E$, respectively. The set of *standard monomials* $\mathrm{SMon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle^E)$ is defined as $\mathrm{Mon}(\mathbb{K}[\underline{x}]^E)$ considered as a subset of $\mathrm{Mon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle^E)$. Abbreviating $\underline{x}^\alpha(e) := x_1^{\alpha_1} \cdots x_n^{\alpha_n}(e)$ for $e \in E$ and $\alpha \in \mathbb{N}^n$, we often write $p \in \mathbb{K}[\underline{\mathrm{x}}]^E$ in multi-index notation $p = \sum_{e,\alpha} p_{e,\alpha}\underline{x}^\alpha(e)$ where $p_{e,\alpha} \in \mathbb{K}$ denotes the coefficient of $\underline{x}^\alpha(e)$.

2. A total order $\prec$ on $\mathrm{Mon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle^E)$ is called a *monomial ordering* if

   (a) $m(e) \prec m'(e')$ implies $pmq(e) \prec pm'q(e')$ for $m, m', p, q \in \mathrm{Mon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle)$ and $e, e' \in E$.

   Similarly, a total order $\prec$ on $\mathrm{SMon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle^E)$ is called a *monomial ordering* if

   (a') $\underline{x}^\alpha(e) \prec \underline{x}^{\alpha'}(e')$ implies $\underline{x}^{\alpha+\gamma}(e) \prec \underline{x}^{\alpha'+\gamma}(e')$ for all $\alpha, \alpha', \gamma \in \mathbb{N}^n$ and $e, e' \in E$.

   A monomial ordering $\prec$ on $\mathrm{(S)Mon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle^E)$ satisfying additionally

   (b) $(e) \preceq m(e)$ for all $m \in \mathrm{(S)Mon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle)$ and $e \in E$

   is called a *(monomial) well-ordering*. Otherwise we say that it is a *(monomial) non-well-ordering*.

3. Let $\prec$ be a monomial ordering on $\mathrm{(S)Mon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle^E)$. If $0 \neq t = \sum_{e\in E, m\in \mathrm{(S)Mon}(\mathbb{K}\langle\underline{\mathrm{x}}\rangle)} t_{e,m} m(e) \in \mathbb{K}\langle\underline{\mathrm{x}}\rangle^E$ with $t_{e,m} \in \mathbb{K}$ and $m'(e') := \max_\prec\{m(e) \mid t_{e,m} \neq 0\}$, then we define

   - $\mathrm{lm}_\prec(t) := m'(e')$, the *leading monomial* of t,
   - $\mathrm{lt}_\prec(t) := t_{e',m'} m'(e')$, the *leading term* of t,
   - $\mathrm{lc}_\prec(t) := t_{e',m'}$, the *leading coefficient* of t,
   - $\mathrm{lcomp}_\prec(t) := e'$, the *leading component* of t,
   - $\mathrm{tail}_\prec(t) := t - \mathrm{lt}_\prec(t)$, the *tail* of t,
   - $\mathrm{le}_\prec(t) := \sum_{1\le j\le k} e_{i_j} \in \mathbb{N}^n$ if $m' = x_{i_1} \cdots x_{i_k}$, the *leading exponent* of t,

- $\mathrm{ele}_\prec(t) := (\mathrm{le}_\prec(t), e')$, the *extended leading exponent* of t,

where $e_{i_j} \in \mathbb{Z}^n$ stands for the i_jth unit vector. For a subset $G \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ we consider the sets

- $\mathrm{l}_\prec(G) := \{\mathrm{ele}_\prec(g) \mid g \in G \setminus \{0\}\} \subseteq \mathbb{N}^n \times E$,
- $\mathrm{L}_\prec(G) := \{\beta + \mathrm{ele}_\prec(g) \mid g \in G \setminus \{0\}, \beta \in \mathbb{N}^n\} \subseteq \mathbb{N}^n \times E$,

where we define $\beta + (\alpha, e) := (\alpha + \beta, e)$ for $\alpha, \beta \in \mathbb{N}^n$ and $e \in E$.

To simplify notation later on, we extend the ordering by setting $\mathrm{lm}_\prec(0) \prec \mathrm{lm}_\prec(t)$ and $\mathrm{lm}_\prec(0) \preceq \mathrm{lm}_\prec(t')$ for all $t, t' \in {}_{\mathbb{K}}\big\langle (\mathrm{S})\mathrm{Mon}(\mathbb{K}\langle \underline{x} \rangle^E) \big\rangle$ with $t \neq 0$. We denote by $\prec$ also the ordering induced by $\prec$ on $\mathbb{N}^n \times E$ via the mapping $(\alpha, e) \mapsto \underline{x}^\alpha(e)$ and adopt an analogous convention for $\mathrm{le}_\prec(0)$ and $\mathrm{ele}_\prec(0)$. Similarly, we define by abuse of notation $\alpha + \mathrm{le}_\prec(0) := \mathrm{le}_\prec(0)$ and $\alpha + \mathrm{ele}_\prec(0) := \mathrm{ele}_\prec(0)$ for any $\alpha \in \mathbb{N}^n$.

We sometimes omit the index $\prec$ if it is clear from the context.

Remark 1.2. Monomial orderings on $\mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle) = \mathrm{Mon}(\mathbb{K}[\underline{x}])$ are the monomial orderings of commutative Gröbner basis theory.

Remark 1.3. Let E be a finite set.

1. Clearly the ordering defined by

$$x_{i_1} \cdots x_{i_k} \prec' x_{j_1} \cdots x_{j_l} \text{ if and only if } k < l \text{ or } k = l \text{ and } (i_1, \ldots, i_k) <_{\mathrm{lex}} (j_1, \ldots, j_k)$$

is a monomial well-ordering on $\mathrm{Mon}(\mathbb{K}\langle \underline{x} \rangle)$. Note that $\underline{x}^{\mathrm{le}(x_{i_1} \cdots x_{i_k})} \preceq x_{i_1} \cdots x_{i_k}$.

2. Refine a monomial ordering $\prec$ on $\mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle^E)$ by a monomial ordering $\prec'$ on $\mathrm{Mon}(\mathbb{K}\langle \underline{x} \rangle^E)$ to a monomial ordering $(\prec, \prec')$ on $\mathrm{Mon}(\mathbb{K}\langle \underline{x} \rangle^E)$ by setting

$$x_{i_1} \cdots x_{i_k}(e) \, (\prec, \prec') \, x_{j_1} \cdots x_{j_l}(e') \text{ if and only if } \mathrm{ele}_{\prec'}(x_{i_1} \cdots x_{i_k}(e)) \prec \mathrm{ele}_{\prec'}(x_{j_1} \cdots x_{j_l}(e')) \text{ or } \mathrm{ele}_{\prec'}(x_{i_1} \cdots x_{i_k}(e)) = \mathrm{ele}_{\prec'}(x_{j_1} \cdots x_{j_l}(e')) \text{ and } x_{i_1} \cdots x_{i_k} \prec' x_{j_1} \cdots x_{j_l}.$$

We denote also by $\prec$ the special ordering $(\prec, \prec')$ with $\prec'$ from Part 1.

3. Let $\prec$ be a monomial ordering on $(\mathrm{S})\mathrm{Mon}(\mathbb{K}\langle \underline{x} \rangle^E)$. Then any $e \in E$ defines a monomial ordering $\prec_e$ on $(\mathrm{S})\mathrm{Mon}(\mathbb{K}\langle \underline{x} \rangle)$ by

$$x_{i_1} \cdots x_{i_k} \prec_e x_{j_1} \cdots x_{j_l} \text{ if and only if } x_{i_1} \cdots x_{i_k}(e) \prec x_{j_1} \cdots x_{j_l}(e).$$

Eventually, we will restrict ourselves to monomial orderings on $\mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle^E)$ and refine them to orderings on $\mathrm{Mon}(\mathbb{K}\langle \underline{x} \rangle^E)$ as outlined in Remark 1.3.2 above if necessary. The following remark lists some of the orderings on $\mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle^E)$ which we will use frequently throughout this article:

Remark 1.4. Let $E_1, \ldots, E_s$ and E be finite sets.

1. An ordering $\prec$ on $\mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle)$ and a total order $<^E$ on E induce the following orderings:

(a) The *term over position ordering* (TOP-ordering) on $\mathrm{SMon}(\mathbb{K}\langle\underline{x}\rangle^E)$:

$$\underline{x}^\alpha(e)\,(<,<^E)\,\underline{x}^\beta(e') \text{ if and only if } \underline{x}^\alpha < \underline{x}^\beta \\ \text{or } \underline{x}^\alpha = \underline{x}^\beta \text{ and } e < e',$$

where $\alpha,\beta \in \mathbb{N}^n$ and $e, e' \in E$.

(b) The *position over term ordering* (POT-ordering) on $\mathrm{SMon}(\mathbb{K}\langle\underline{x}\rangle^E)$:

$$\underline{x}^\alpha(e)\,(<^E,<)\,\underline{x}^\beta(e') \text{ if and only if } e < e' \\ \text{or } e = e' \text{ and } \underline{x}^\alpha < \underline{x}^\beta,$$

where $\alpha,\beta \in \mathbb{N}^n$ and $e, e' \in E$.

These orderings are well-orderings if and only if $<$ is a well-ordering.

2. Orderings $<_1^{E_1},\dots,<_s^{E_s}$ on $\mathrm{SMon}(\mathbb{K}\langle\underline{x}\rangle^{E_1}),\dots,\mathrm{SMon}(\mathbb{K}\langle\underline{x}\rangle^{E_s})$, respectively, define the *(module) block ordering* $(<_1^{E_1},\dots,<_s^{E_s})$ on $\mathrm{SMon}(\mathbb{K}\langle\underline{x}\rangle^{E_1\sqcup\cdots\sqcup E_S})$ by

$$\underline{x}^\alpha(e)\,(<_1^{E_1},\dots,<_s^{E_s})\,\underline{x}^\beta(e') \text{ if and only if, for } e \in E_i, e' \in E_j,\ i > j \\ \text{or } i = j \text{ and } \underline{x}^\alpha(e) <_i^{E_i} \underline{x}^\beta(e'),$$

where $\alpha,\beta \in \mathbb{N}^n$. Notice that $(<_1^{E_1},\dots,<_s^{E_s})$ is a well-ordering if and only if all $<_i^{E_i}$ are well-orderings.

Convention 1.5. Let $E_1,\dots,E_s$ and E be finite sets. If we write from now on $<^E$, we implicitly assume that $<^E$ is some ordering on $\mathrm{SMon}(\mathbb{K}\langle\underline{x}\rangle^E)$. Similarly, $(<_1^{E_1},\dots,<_s^{E_s})$ always denotes a block ordering on $\mathrm{SMon}(\mathbb{K}\langle\underline{x}\rangle^{E_1\sqcup\cdots\sqcup E_s})$.

We identify $\mathbb{K}\langle\underline{x}\rangle^{E_1}\oplus\cdots\oplus\mathbb{K}\langle\underline{x}\rangle^{E_s} \cong \mathbb{K}\langle\underline{x}\rangle^{E_1\sqcup\cdots\sqcup E_s}$ to define the set of (standard) monomials of the former module as well as monomial orderings on them.

The following notions rely on the work by Bergman (1978):

Definition 1.6. Let E be a finite set, $<$ a monomial ordering on $\mathrm{Mon}(\mathbb{K}\langle\underline{x}\rangle^E)$. For $s \in \mathbb{K}\langle\underline{x}\rangle^E \setminus\{0\}$ and $m, m' \in \mathrm{Mon}(\mathbb{K}\langle\underline{x}\rangle)$ we define the $\mathbb{K}$-linear *reduction (map) (with respect to $(S,<)$)*

$$\rho_{m,s,m'} : \mathbb{K}\langle\underline{x}\rangle^E \to \mathbb{K}\langle\underline{x}\rangle^E,\ x_{i_1}\cdots x_{i_l}(e) \mapsto \begin{cases} m(-\frac{1}{\mathrm{lc}_<(s)}\,\mathrm{tail}_<(s))m' & \text{if } x_{i_1}\cdots x_{i_l}(e) = m\,\mathrm{lm}_<(s)m', \\ x_{i_1}\cdots x_{i_l}(e) & \text{otherwise.}\end{cases}$$

Let $S \subseteq \mathbb{K}\langle\underline{x}\rangle^E$ be a subset. A finite composition of such maps with $s \in S$ is called a *reduction sequence* (with respect to $(S,<)$).

1. For a reduction sequence ρ with respect to $(S,<)$ we say that $t \in \mathbb{K}\langle\underline{x}\rangle^E$ reduces to $\rho(t)$, a *reduction* of t (with respect to $(S,<)$).

2. We call $t \in \mathbb{K}\langle\underline{x}\rangle^E$ *irreducible* (with respect to $(S,<)$) if $\rho(t) = t$ for all reductions ρ. Such elements generate the $\mathbb{K}$-submodule $(\mathbb{K}\langle\underline{x}\rangle^E)^{\mathrm{irr}}_{S,<} \subseteq \mathbb{K}\langle\underline{x}\rangle^E$.

3. Suppose that $\prec$ is a special ordering as in Remark 1.3.2.Then we call $(S, \prec)$ a *commutation system* if $S = \{s_{i,j,e} \mid 1 \leq i < j \leq n, e \in E\}$ where

$$s_{i,j,e} := x_j x_i(e) - c_{ij} x_i x_j(e) - d_{ij} \text{ with } \operatorname{lm}_\prec(d_{ij}) \prec x_i x_j(e) \prec x_j x_i(e),$$

$c_{ij} \in \mathbb{K}^*$ and $d_{ij} \in \mathbb{K}[\underline{x}]^E$, and if every element in $\mathbb{K}\langle\underline{x}\rangle^E$ reduces to an element in $\mathbb{K}[\underline{x}]^E$. We refer to the elements of S as commutation relations and to $\rho_{m,s_{i,j,e},m'}$ as a *commutation reduction*.

We say that an ordering $\prec'$ is *compatible* with $(S, \prec)$ if $(S, \prec')$ is a commutation system.

Remark 1.7. Let E be a finite set, $S \subseteq \mathbb{K}\langle\underline{x}\rangle^E$ and $\prec$ a monomial ordering.

1. If $(S, \prec)$ is a commutation system and $S \subseteq S' \subseteq \mathbb{K}\langle\underline{x}\rangle^E$, then $(\mathbb{K}\langle\underline{x}\rangle^E)^{\text{irr}}_{S',\prec} \subseteq \mathbb{K}[\underline{x}]^E = (\mathbb{K}\langle\underline{x}\rangle^E)^{\text{irr}}_{S,\prec}$.

2. If $\prec$ is a well-ordering, every element of $\mathbb{K}\langle\underline{x}\rangle^E$ reduces to an irreducible element with respect to $(S, \prec)$. If in addition S is a two-sided submodule, then this reduction is unique. This identifies $(\mathbb{K}\langle\underline{x}\rangle^E)^{\text{irr}}_{S,\prec} = \mathbb{K}\langle\underline{x}\rangle^E / S$ as $\mathbb{K}$-vector spaces and we write

$$\rho_{S,\prec} : \mathbb{K}\langle\underline{x}\rangle^E \to (\mathbb{K}\langle\underline{x}\rangle^E)^{\text{irr}}_{S,\prec}$$

for the map sending elements to their unique irreducible reduction.

3. If $(S, \prec)$ is a commutation system and $\prec$ a well-ordering, we can determine for an element $p \in \mathbb{K}\langle\underline{x}\rangle$ a finite set $U \subseteq \mathbb{K}\langle\underline{x}\rangle \times S \times \mathbb{K}\langle\underline{x}\rangle$ such that

$$p = \rho_{S,\prec'}(p) + \sum_{(t,s,t')\in U} tst' \text{ with } \operatorname{lm}_{\prec'}(\rho_{S,\prec'}(p)) = \operatorname{lm}_{\prec'}(p) \text{ and } \operatorname{lm}_{\prec'}(tst') \preceq' \operatorname{lm}_{\prec'}(p)$$

for all orderings $\prec'$ compatible with $(S, \prec)$. In particular $\rho_{S,\prec} = \rho_{S,\prec'}$.

We are particularly interested in the following class of $\mathbb{K}$-algebras:

Definition 1.8. A *PBW-reduction-datum* $(\mathbb{K}\langle\underline{x}\rangle, S, I, \prec)$ consists of a commutation system $(S, \prec)$, where $\prec$ is a well-ordering and $I \subseteq \mathbb{K}[\underline{x}]$ a finite set such that

$$\mathrm{L}_\prec(I) = \mathrm{l}_\prec(\langle I \cup S\rangle \cap \mathbb{K}[\underline{x}]). \tag{1}$$

It defines a $\mathbb{K}$-algebra $\mathbb{K}\langle\underline{x}\rangle / \langle I \cup S\rangle$, which we call a *PBW-reduction-algebra* and write by abuse of notation

$$\mathbb{K}\langle\underline{x}\rangle / \langle I \cup S\rangle = (\mathbb{K}\langle\underline{x}\rangle, S, I, \prec).$$

We say that a monomial (well-)ordering $\prec'$ is a (monomial) (well-)ordering on $\mathbb{K}\langle\underline{x}\rangle / \langle I \cup S\rangle$ if $\prec'$ is compatible with $(S, \prec)$.

Remark 1.9. For a commutation system $(S, \prec)$ and $I \subseteq \mathbb{K}[\underline{x}]$ we have that

$$\mathrm{l}_\prec(\langle I \cup S\rangle \cap \mathbb{K}[\underline{x}]) = \mathrm{L}_\prec(\langle I \cup S\rangle \cap \mathbb{K}[\underline{x}]).$$

Indeed, $\rho_{S,\prec}(\underline{x}^\alpha r) \in \langle I \cup S\rangle \cap \mathbb{K}[\underline{x}]$ with $\operatorname{lm}_\prec(\rho_{S,\prec}(\underline{x}^\alpha r)) = \alpha + \operatorname{lm}_\prec(r)$ for $r \in \langle I \cup S\rangle \cap \mathbb{K}[\underline{x}]$ and $\alpha \in \mathbb{N}^n$. In particular the inclusion $\subseteq$ in 1.8(1) is always satisfied. This makes Condition 1.8(1) equivalent to

$$\mathrm{L}_\prec(I) = \mathrm{L}_\prec(\langle I \cup S\rangle \cap \mathbb{K}[\underline{x}]).$$

Lemma 1.10. *For a PBW-reduction-datum* $(\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)$ *we can identify the* $\mathbb{K}$*-vector spaces*

$$\mathbb{K}\langle \underline{x}\rangle \,/\, \langle I \cup S\,\rangle = \mathbb{K}\langle \underline{x}\rangle^{\mathrm{irr}}_{\langle I\cup S\,\rangle,\prec} = \bigoplus_{\alpha\notin \mathrm{L}_{\prec}(I)} \mathbb{K}\,\underline{x}^{\alpha}$$

Proof. The first equality is due to Remark 1.7.2. Since the set of irreducible standard monomials with respect to $(\langle I \cup S\,\rangle, \prec)$ forms a $\mathbb{K}$-basis of $\mathbb{K}\langle \underline{x}\rangle^{\mathrm{irr}}_{\langle I\cup S\,\rangle,\prec}$ by Remark 1.7.1, it suffices to show that this set agrees with $\{\underline{x}^{\alpha} \mid \alpha \notin \mathrm{L}_{\prec}(I)\}$. Clearly, the former set is contained in the latter. On the other hand, $\underline{x}^{\alpha}$ with $\alpha \notin \mathrm{L}_{\prec}(I) = \mathrm{l}_{\prec}(\langle S \cup I\rangle \cap \mathbb{K}[\underline{x}])$ is indeed irreducible. ☐

Proposition 1.11. *PBW-algebras are precisely the PBW-reduction-algebras with PBW-reduction datum of type* $(\mathbb{K}\langle \underline{x}\rangle, S, \{0\}, \prec)$*. In particular, polynomial rings and Weyl algebras are PBW-reduction-algebras.*

Proof. Given a PBW-algebra with commutation system $(S, \prec)$ with respect to the well-ordering $\prec$. Then $\langle S\,\rangle \cap \mathbb{K}[\underline{x}] = \{0\}$ by definition and hence Equation (1) is satisfied with $I = \{0\}$. The converse is due to Lemma 1.10. ☐

Lemma 1.12. *Consider the commutation system* $(S, \prec)$*, where* $\prec$ *is a well-ordering, the finite set* $I \subseteq \mathbb{K}[\underline{x}]$ *and the two-sided ideal* $R \subseteq \mathbb{K}\langle \underline{x}\rangle$ *containing* S *and* I *and satisfying* $\mathrm{L}_{\prec}(I) = \mathrm{l}_{\prec}(R \cap \mathbb{K}[\underline{x}])$*. For* $p \in \mathbb{K}\langle \underline{x}\rangle$ *one can compute* $a \in \mathbb{K}\langle \underline{x}\rangle^{I}$ *and a finite set* $U \subseteq \mathbb{K}\langle \underline{x}\rangle \times S \times \mathbb{K}\langle \underline{x}\rangle$ *such that*

$$p = \rho_{R,\prec}(p) + \sum_{g\in I} a_g g + \sum_{(t,s,t')\in U} tst',$$

$\mathrm{le}(a_g) + \mathrm{le}(g) \leq \mathrm{le}(p)$ *with equality for some* $g \in I$ *and* $\mathrm{le}(t) + \mathrm{le}(s) + \mathrm{le}(t') \leq \mathrm{le}(p)$*. In particular,* $R = \langle S \cup I\rangle$ *and* $(\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)$ *is a PBW-reduction datum.*

Proof. By Remark 1.7.3 we can write $p \in \mathbb{K}\langle \underline{x}\rangle$ as

$$p = \rho_{S,\prec}(p) + \sum_{(t,s,t')\in U_p} tst' \tag{2}$$

for some $U_p \subseteq \mathbb{K}\langle \underline{x}\rangle \times S \times \mathbb{K}\langle \underline{x}\rangle$ with $\mathrm{le}(t) + \mathrm{le}(s) + \mathrm{le}(t') \leq \mathrm{le}(p)$. If $\mathrm{le}(\rho_{S,\prec}(p)) \notin \mathrm{l}_{\prec}(R \cap \mathbb{K}[\underline{x}])$, then $\mathrm{lm}(\rho_{S,\prec}(p)) \in \mathbb{K}\langle \underline{x}\rangle^{\mathrm{irr}}_{R,\prec}$ and we continue with $\mathrm{tail}(\rho_{S,\prec}(p))$. Otherwise pick $g \in I$ such that $\mathrm{le}(p) = \mathrm{le}(\rho_{S,\prec}(p)) = \mathrm{le}(g) + \alpha$. Remark 1.7.3 also yields an Equation (2) with p replaced by $\underline{x}^{\alpha} g$. Hence we obtain

$$p = c\underline{x}^{\alpha} g + (\mathrm{tail}(\rho_{S,\prec}(p)) - \mathrm{tail}(c\rho_{S,\prec}(\underline{x}^{\alpha} g))) + \sum_{(t,s,t')\in U_p} tst' - c \sum_{(t,s,t')\in U_{\underline{x}^{\alpha} g}} tst'$$

for $c = \mathrm{lc}(\rho_{S,\prec}(p))/\,\mathrm{lc}(\rho_{S,\prec}(\underline{x}^{\alpha} g))$. Induction on the well-ordering $\prec$ finishes the proof. ☐

Proposition 1.13. *For a commutation system* $(S, \prec)$ *with well-ordering* $\prec$ *and a two-sided ideal* $S \subseteq R \subseteq \mathbb{K}\langle \underline{x}\rangle$ *exists a finite set* $I \subseteq \mathbb{K}[\underline{x}]$ *such that* $\mathbb{K}\langle \underline{x}\rangle\,/R = (\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)$ *is a PBW-reduction-algebra.*

Proof. Consider the set

$$L := \mathrm{L}_{\prec}(R \cap \mathbb{K}[\underline{x}]) \subseteq \mathbb{N}^n\,.$$

By Dickson's Lemma there is a finite subset $L' \subseteq L$ such that for every $\alpha \in L$ exists an $\alpha' \in L'$ with $\alpha \in \mathbb{N}^n + \alpha'$. Choose for every $\alpha' \in L'$ an $r_{\alpha'} \in R \cap \mathbb{K}[\underline{x}]$ having leading exponent α'. Setting

$$I := \{r_{\alpha'} \mid \alpha' \in L'\},$$

$\mathbb{K}\langle \underline{x} \rangle / R = (\mathbb{K}\langle \underline{x} \rangle, S, I, \prec)$ is a PBW-reduction-algebra by Lemma 1.12. □

In general it is unclear how to obtain the set I of the PBW-reduction datum. In the following special case this is possible.

Lemma 1.14. *Consider the* $\mathbb{K}$*-algebra* $\mathbb{K}\langle \underline{x}, \underline{y} \rangle := \mathbb{K}\langle x_1, \dots, x_n, y_1, \dots, y_m \rangle$*, an ideal* $I \subseteq \mathbb{K}[\underline{x}]$ *and a commutation system* $(S, \prec)$ *such that*

$$S = \{[x_j, x_i] \mid 1 \le i < j \le n\} \cup \{[y_l, y_k] - d_{kl} \mid 1 \le k < l \le m\} \cup \{[y_k, x_i] - f_{ik} \mid 1 \le i \le n, 1 \le k \le m\},$$

where $d_{kl}, f_{ik} \in \mathbb{K}[\underline{x}]$*. Suppose the surjective* $\mathbb{K}$*-linear homomorphism*

$$\psi : \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[\underline{x}]/J)\underline{y}^\beta \to \mathbb{K}\langle \underline{x}, \underline{y} \rangle / \langle J \cup S \rangle, \ \overline{\underline{x}^\alpha} \underline{y}^\beta \mapsto \overline{\underline{x}^\alpha \underline{y}^\beta}$$

is injective and that I' *is a Gröbner basis of* $I \subseteq \mathbb{K}[\underline{x}]$ *with respect to the ordering induced by* $\prec$*. Then* $(\mathbb{K}\langle \underline{x}, \underline{y} \rangle, S, I', \prec)$ *is a PBW-reduction datum.*

Proof. Let $0 \neq p = \sum_{\beta \in \mathbb{N}^m} p_\beta \underline{y}^\beta \in \langle I \cup S \rangle \cap \mathbb{K}[\underline{x}, \underline{y}]$ with $p_\beta \in \mathbb{K}[\underline{x}]$. Then $\overline{0} = \overline{p} \in \mathbb{K}\langle \underline{x}, \underline{y} \rangle / \langle I \cup S \rangle$, hence $\sum_{\beta \in \mathbb{N}^m} \overline{p_\beta} \underline{y}^\beta = \psi^{-1}(\overline{p}) = 0$ by hypothesis and $p_\beta \in I$ for all $\beta \in \mathbb{N}^m$. Since I' is a Gröbner basis of $I \subseteq \mathbb{K}[\underline{x}]$ with respect to the ordering induced by $\prec$, it follows that

$$(\alpha', \beta') := \mathrm{le}_\prec(p) = (\mathrm{le}_\prec(p_{\beta'}), \beta') \in L_\prec(I'). \qquad \square$$

Remark 1.15. For S as in Lemma 1.14, any special well-ordering $\prec$ as in Definition 1.6(3) that satisfies

$$\underline{x}^\alpha \underline{y}^\beta \prec \underline{x}^{\alpha'} \underline{y}^{\beta'} \text{ if } |\beta| < |\beta'|$$

(with $\alpha, \alpha' \in \mathbb{N}^n$ and $\beta, \beta' \in \mathbb{N}^m$) makes $(S, \prec)$ a commutation system.

Definition 1.16. In the situation of Lemma 1.14 we call $A = (\mathbb{K}\langle \underline{x}, \underline{y} \rangle, S, I', \prec)$ an *elementary PBW-reduction datum / algebra*.

Generalizing Proposition 1.11, the following example describes differential operators on smooth, complex affine varieties as PBW-reduction-algebras.

Example 1.17. Let X be a smooth irreducible complex affine variety of dimension m defined by the prime ideal $I \subseteq \mathbb{C}[\underline{x}]$. Its tangent sheaf Θ_X is a locally free O_X-module. Note that every element of $\Theta_X(X) = \mathrm{Der}_\mathbb{C}(\mathbb{C}[\underline{x}]/I)$ is of the form $\overline{\theta}$ for some $\theta \in \Theta_{\mathbb{C}^n}(\mathbb{C}^n) = \mathrm{Der}_\mathbb{C}(\mathbb{C}[\underline{x}])$ with $\theta(I) \subseteq I$. After shrinking X if necessary there is an O_X-basis $\overline{\theta_1}, \dots, \overline{\theta_m} \in \Theta_X(X) = \mathrm{Der}_\mathbb{C}(\mathbb{C}[\underline{x}]/I)$ of Θ_X and there are regular functions $\overline{f_1}, \dots, \overline{f_m} \in \mathbb{C}[\underline{x}]/I$ satisfying $[\overline{\theta_i}, \overline{\theta_j}] = 0$ and $[\overline{\theta_i}, \overline{f_j}] = \delta_{ij}$ for $1 \le i, j \le m$.

1. The global sections of the sheaf of differential operators $\mathcal{D}_X$ form an elementary PBW-reduction-algebra: We have a $\mathbb{C}$-linear isomorphism (see (Rottner, 2018, Lemma 1.2.7))

$$\phi : \bigoplus_{\beta\in\mathbb{N}^m}(\mathbb{C}[\underline{x}]/I)\underline{y}^\beta \to \mathcal{D}_X(X) = \mathbb{C}\langle\overline{x_1},\dots,\overline{x_n},\overline{\theta_1},\dots,\overline{\theta_m}\rangle \subseteq \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\underline{x}]/I),$$

$$\overline{\underline{x}^\alpha}\underline{y}^\beta \mapsto \overline{x_1}^{\alpha_1}\cdots\overline{x_n}^{\alpha_n}\overline{\theta_1}^{\beta_1}\cdots\overline{\theta_m}^{\beta_m}$$

and the generators of the $\mathbb{C}$-algebra $\mathcal{D}_X(X)$ satisfy $[\overline{x_j},\overline{x_i}]=0$, $[\overline{\theta_p},\overline{\theta_k}]=0$ and $[\overline{\theta_k},\overline{x_i}]=\overline{\theta_k(x_i)}$ for $1\le i\le j\le n$ and $1\le k\le p\le m$. Consequently, ψ factors through the algebra

$$T_X := \mathbb{C}\langle\underline{x},\underline{y}\rangle/\langle S\cup I\rangle,$$

where

$$\begin{aligned} S :=&\{[x_j,x_i] \mid 1\le i<j\le n\}\cup\{[y_p,y_k] \mid 1\le k<p\le m\}\\ &\cup\{[y_k,x_i]-\theta_k(x_i) \mid 1\le i\le n, 1\le k\le m\}. \end{aligned}$$

This leads to isomorphisms

$$\begin{array}{ccccc} \phi : \bigoplus_{\beta\in\mathbb{N}^m}(\mathbb{C}[\underline{x}]/I)\underline{y}^\beta & \xrightarrow[\cong]{\psi} & T_X & \xrightarrow[\cong]{} & \mathcal{D}_X(X) \\ \overline{\underline{x}^\alpha}\underline{y}^\beta & \longmapsto & \overline{\underline{x}^\alpha\underline{y}^\beta} & \longmapsto & \overline{x_1}^{\alpha_1}\cdots\overline{x_n}^{\alpha_n}\theta_1^{\beta_1}\cdots\theta_m^{\beta_m}. \end{array} \tag{3}$$

identifying $\mathcal{D}_X(X)$ with the elementary PBW-reduction-algebra T_X.

2. By identifying X with the closed subvariety $V(I, t-f_m)\subseteq\mathbb{C}^n\times\mathbb{C}_t$, we may assume f_m agrees with x_n, and that $\theta_i(x_n)=\delta_{i,m}$. Consider the obvious algebra homomorphism $\mathbb{C}\langle\underline{x},y_1,\dots,y_{m-1},z\rangle\to\mathcal{D}_X(X)$ sending z to $\overline{x_n\theta_m}$ with image V. It factors through the PBW-reduction-algebra $T_X^V=\mathbb{C}\langle\underline{x},y_1,\dots,y_{m-1},z\rangle/\langle S_V\cup I\rangle$ where

$$\begin{aligned} S_V := \{&[x_j,x_i],[y_l,y_k],[z,y_k],[y_k,x_i]-\theta_k^l(x_i),[z,x_i]-x_n\theta_m^l(x_i) \mid\\ &1\le i\le j\le n, 1\le k\le l\le m-1\}\setminus\{0\}. \end{aligned}$$

This extends Equation (3) to a commutative diagram of $\mathbb{C}$-linear maps

$$\begin{array}{ccccc} \bigoplus_{\beta\in\mathbb{N}^m}(\mathbb{C}[\underline{x}]/I)\underline{y}^\beta & \xrightarrow[\cong]{\psi} & T_X & \xrightarrow[\cong]{} & \mathcal{D}_X(X) \\ \uparrow & & \uparrow & & \uparrow \\ \bigoplus_{\beta\in\mathbb{N}^{m-1},\gamma\in\mathbb{N}}(\mathbb{C}[\underline{x}]/I)y_1^{\beta_1}\cdots y_{m-1}^{\beta_{m-1}}z^\gamma & \twoheadrightarrow & T_X^V & \twoheadrightarrow & V \end{array}$$

where the right hand square consists of $\mathbb{C}$-algebra homomorphisms. One can show that the left vertical map is injective. It follows that the bottom maps are isomorphisms and the vertical maps are injections. So we may identify V with the elementary PBW-reduction-algebra T_X^V.

3. Let $\phi_{x_n} : \mathbb{C}\langle \underline{x}, y_1, \ldots, y_{m-1}, z\rangle \to \mathbb{C}\langle x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}, z\rangle$ be the $\mathbb{C}$-algebra homomorphism that maps x_n to 0 and acts on all other variables as identity. In the situation of Part 2, V/x_nV can be realized as the elementary PBW-reduction-algebra

$$T_X^{V/x_nV} := (\mathbb{C}\langle x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}, z\rangle, S_{V/x_nV}, I_{V/x_nV}, \prec^{V/x_nV})$$

defined as follows: For

$$S_{V/x_nV} := \{[x_j, x_i], [y_p, y_k], [z, y_k], [z, x_i], [y_k, x_i] - \phi_{x_n}(\theta_k^l(x_i)) \mid \\ 1 \le i \le j \le n-1, 1 \le k \le p \le m-1\} \setminus \{0\},$$

pick a well-ordering $\prec^{V/x_nV}$ as in Remark 1.15. Now let $I_{V/x_nV} \subseteq \mathbb{C}[x_1, \ldots, x_{m-1}]$ be a Gröbner basis of $\phi_{x_n}(I)$ with respect to the ordering induced by $\prec^{V/x_nV}$. Note that the canonical projection $V \to V/x_nV$ induces the same map $T_X^V \to T_X^{V/x_nV}$ as ϕ_{x_n}.

4. With the assumption of Part 2 assume that the subvariety $X_0 := V(x_n) \cap X \subseteq X$ is smooth. Then $(\overline{f_i}, \overline{\theta_i})_{1 \le i \le m-1}$ is a global coordinate system on X_0. By Part 1 $\mathcal{D}_{X_0}(X_0)$ identifies with the PBW-reduction-algebra T_{X_0}, whose commutation system is obtained by deleting all equations involving y_m from S. The natural isomorphism $V/x_nV \cong \mathcal{D}_{X_0}(X_0)[\overline{x_n\theta_m}]$ then identifies with the isomorphism $T_X^{V/x_nV} \cong T_{X_0}[z]$ that is induced by the identity of $\mathbb{C}\langle x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}, z\rangle$.

Remark 1.18. There is an algorithmic approach to the sheaf of differential operators on smooth affine varieties by Oaku (1996). Consider the setup of Example 1.17. Oaku suggests two methods: The first one is based on the statement that the $\mathbb{C}$-subalgebra of the Weyl-algebra generated by $x_1, \ldots, x_n$ and $\theta_1, \ldots, \theta_m$ equals $\bigoplus_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m} \mathbb{C}\, \underline{x}^\alpha \theta_1^{\beta_1} \cdots \theta_m^{\beta_m}$. This does not hold in general: Indeed, there is always a local coordinate system such that $f_i = x_i$ and $\theta_i = \partial_i + \sum_{m+1 \le k \le n} a_k^i(\underline{x})\partial_k$ with $a_k^i(\underline{x}) \in \mathbb{C}[\underline{x}]$. In this case $[\theta_j, \theta_i] \in \sum_{m+1 \le k \le n} \mathbb{C}[\underline{x}]\partial_k$ can only be contained in the above direct sum if it is 0. However, $\theta_1, \ldots, \theta_m$ do not commute in general.

Oaku's second method uses the Leibnitz rule to define a non-associative "multiplication". The resulting algorithm is essentially equivalent to Algorithm 1.31. However, Oaku's proof of correctness relies again on the above false statement.

Proposition 1.19. *PBW-reduction-algebras are left and right Noetherian rings.*

Proof. Let $A = (\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)$ be a PBW-reduction-algebra with $S = \{x_jx_i - c_{ij}x_ix_j - d_{ij} \mid 1 \le i < j \le n\}$. This gives rise to a multi-filtration $F_\bullet^\prec$ on A indexed by $\mathbb{N}^n$ (see Gómez-Torrecillas and Lobillo (2000)) given by

$$F_\alpha^\prec A := \sum_{\underline{x}^\beta \preceq \underline{x}^\alpha} \mathbb{K}\,\overline{\underline{x}^\beta}, \quad F_{\prec\alpha}^\prec A := \bigcup_{\beta \in \mathbb{N}^n : \underline{x}^\beta \prec \underline{x}^\alpha} F_\beta^\prec A = \sum_{\underline{x}^\beta \prec \underline{x}^\alpha} \mathbb{K}\,\overline{\underline{x}^\beta}$$

for $\alpha \in \mathbb{N}^n$. Note that this filtration is exhaustive by 1.10. By (Gómez-Torrecillas and Lobillo, 2000, Lemma 1.2) it suffices to proof the claim for the associated multi-graded algebra

$$\mathrm{Gr}^{F^\prec} A := \bigoplus_{\alpha \in \mathbb{N}^n} F_\alpha^\prec A / F_{\prec\alpha}^\prec A.$$

It identifies with a factor algebra of a PBW-algebra by the isomorphism

$$\varphi : \mathrm{Gr}^{F^\prec} A \to \left(\mathbb{K}\langle \underline{x}\rangle \,/\, \left\langle \{x_jx_i - c_{ij}x_ix_j \mid 1 \le i < j \le n\}\right\rangle\right) / \left\langle \{\overline{\mathrm{lm}_\prec(p)} \mid p \in I\}\right\rangle,$$
$$\mathrm{Gr}_{e_i}^{F^\prec} A \ni \overline{x_i} \mapsto \overline{x_i} + \left\langle \{\overline{\mathrm{lm}_\prec(p)} \mid p \in I\}\right\rangle.$$

The latter is left and right Noetherian (see e.g. (Bueso et al., 2001, Theorem 4.1)). ☐

Definition 1.20. Let A be a PBW-reduction-algebra and E a finite set.

1. If $(\mathbb{K}\langle \underline{x}\rangle, S_e, I_e, \prec_e)$ is a PBW-reduction datum of A for $e \in E$, we call $(\mathbb{K}\langle \underline{x}\rangle, S_e, I_e, \prec_e)_{e\in E}$ a PBW-reduction datum for A^E and write $A^E = (\mathbb{K}\langle \underline{x}\rangle, S_e, I_e, \prec_e)_{e\in E}$. We say that the monomial (well-)ordering $\prec^E$ on $\mathrm{SMon}(\mathbb{K}\langle \underline{x}\rangle^E)$ is a *(well-)ordering* on A^E if $\prec^E_e$ is an ordering on eth summand $(\mathbb{K}\langle \underline{x}\rangle, S_e, I_e, \prec_e)$ of A^E. If $\prec^E_e = \prec_e$ for all $e \in E$, we call $(\mathbb{K}\langle \underline{x}\rangle, S_e, I_e, \prec_e)_{e\in E}$ a PBW-reduction datum for $(A^E, \prec^E)$ and write $(A^E, \prec^E) = (\mathbb{K}\langle \underline{x}\rangle, S_e, I_e, \prec_e)_{e\in E}$.

2. Abusing notation, we set for $(A^E, \prec^E) = (\mathbb{K}\langle \underline{x}\rangle, S_e, I_e, \prec_e)_{e\in E}$

$$\rho_{A^E,\prec^E} := \bigoplus_{e\in E} \rho_{\langle I_e \cup S_e\rangle, \prec^E_e} : \mathbb{K}\langle \underline{x}\rangle^E \to (\mathbb{K}\langle \underline{x}\rangle^E)^{\mathrm{irr}}_{A^E,\prec^E} := \bigoplus_{e\in E} \mathbb{K}\langle \underline{x}\rangle^{\mathrm{irr}}_{\langle I_e\cup S_e\rangle, \prec^E_e}(e).$$

We also define the map

$$\tau_{A^E,\prec^E} : A^E \to (\mathbb{K}\langle \underline{x}\rangle^E)^{\mathrm{irr}}_{A^E,\prec^E} \subseteq \mathbb{K}\langle \underline{x}\rangle^E$$

as the inverse of the composed map $(\mathbb{K}\langle \underline{x}\rangle^E)^{\mathrm{irr}}_{A^E,\prec^E} \hookrightarrow \mathbb{K}\langle \underline{x}\rangle^E \twoheadrightarrow A^E$ using Remark 1.7.3. We sometimes also use the notation $\rho_{\prec^E}$ and $\tau_{\prec^E}$ for the above maps if that does not cause any ambiguity.

For $0 \neq a \in A$, we define the data introduced in Definition 1.1(3) by the corresponding data of $\tau_{(A^E,\prec^E)}(a)$ and adapt the convention for the leading exponents and monomials of 0 accordingly.

Remark 1.21. Let $A = (\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)$ be a PBW-reduction-algebra and E and $E_1, \ldots, E_s$ finite sets. Then we have:

1. Given a total order $<^E$ on E PBW-reduction data for $(A^E, (\prec, <^E))$ and $(A^E, (<^E, \prec))$ are given by $(\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)_{e\in E}$.

2. We identify $A^{E_1\sqcup\cdots\sqcup E_s} = A^{E_1} \oplus \cdots \oplus A^{E_s}$ extending the notions of Definition 1.20 to the latter. PBW-reduction data on $(A^{E_i}, \prec^{E_i})$ define PBW-reduction data on $(A^{E_1} \oplus \cdots \oplus A^{E_s}, \prec^{E_1,\ldots,E_s}_{1,\ldots,s})$.

Definition 1.22. Let A be a PBW-reduction-algebra, E a finite set, $(A^E, \prec^E) = (\mathbb{K}\langle \underline{x}\rangle, S_e, I_e, \prec_e)_{e\in E}$ and $M \subseteq A^E$ an A-submodule.

1. We call the finite set $G \subseteq M$ a *Gröbner basis* of M (with respect to $\prec^E$) if every $m \in M$ has a so-called *standard representation*, i.e., there exists $a \in A^G$ such that

$$m = \sum_{g\in G} a_g g \text{ and } \mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(a_g) + \mathrm{ele}_{\prec^E}(g) \preceq^E \mathrm{ele}_{\prec^E}(m) \text{ for all } g \in G.$$

2. If G is a Gröbner basis of M, we say that G is reduced if $0 \notin G$, $\mathrm{lc}_{\prec^E}(g) = 1$ for all $g \in G$, and if we have for all $g \in G$, $e \in E$ and $\alpha \in \mathbb{N}^n$

$$(\tau_{A^E,\prec^E}(g))_{e,\alpha} \neq 0 \text{ implies } (\alpha, e) \neq \mathrm{ele}_{\prec^E}(g') + \gamma \text{ for all } g \neq g' \in G, \gamma \in \mathbb{N}^n.$$

Remark 1.23. Let A be a PBW-reduction-algebra, E a finite set, $<^E$ an ordering on $A^E = (\mathbb{K}\langle \underline{x} \rangle, S_e, I_e, <_e)_{e\in E}$ and $M \subseteq A^E$ an A-submodule. To circumvent the problem that we do in general not have a well-defined notion of leading exponents of elements of A^E with respect to $<^E$, we define Gröbner bases in this situation as follows: We say that a finite set $G \subseteq M$ is a *Gröbner basis* of M with respect to $<^E$ if there exists $h \in (\mathbb{K}[\underline{x}]^E)^G$ with $\overline{h_g} = g$ for $g \in G$ such that for every $t \in \mathbb{K}[\underline{x}]^E$ with $\bar{t} \in M$ exists $a \in \mathbb{K}[\underline{x}]^G$ with

$$\bar{t} = \sum_{g\in G} \overline{a_g} g \text{ and } \mathrm{le}_{<^E_{\mathrm{lcomp}(h_g)}}(a_g) + \mathrm{ele}_{<^E}(h_g) \preceq^E \mathrm{ele}_{<_E}(t) \text{ for all } g \in G.$$

We say in this case that $\{h_g \mid g \in G\}$ induces a Gröbner basis of M (with respect to $<^E$).

Notice that since there exists by definition of PBW-reduction-algebras a well-ordering on A, every $m \in M$ has a representative in $\mathbb{K}[\underline{x}]^E$. Moreover, this definition is compatible with Definition 1.22(1).

Definition 1.24. Let A be a PBW-reduction-algebra, E a finite set, $(A^E, <^E) = (\mathbb{K}\langle \underline{x} \rangle, S_e, I_e, <_e)_{e\in E}$ and let $a, a' \in A^E$ be nonzero.

1. Given a finite set $G \subseteq A^E$, we call $r \in A^E$ a *(left) normal form* of a with respect to G if
   (a) there exists some $h \in A^G$ with
   $$a = \sum_{g\in G} h_g g + r$$
   such that $\mathrm{le}_{<^E_{\mathrm{lcomp}(g)}}(h_g) + \mathrm{ele}_{<^E}(g) \preceq^E \mathrm{ele}_{<^E}(a)$ for all $g \in G$ and
   (b) $\mathrm{ele}_{<^E}(r) \notin L_{<^E}(G)$ if $r \neq 0$.

   We call r *reduced* if $(\alpha, e) \notin L_{<^E}(G)$ given that $(\tau_{(A^E,<^E)}(r))_{e,\alpha} \neq 0$. We define the normal form of $0 \in A^E$ with respect to G to be 0.

2. The *s-polynomial* of a and a' with $e := \mathrm{lcomp}(a) = \mathrm{lcomp}(a')$ is defined by
   $$\mathrm{spoly}(a, a') := \begin{cases} \frac{1}{\mathrm{lc}(\underline{x}^{c_{a,a'}} a)} \underline{x}^{c_{a,a'}} a - \frac{1}{\mathrm{lc}(\underline{x}^{c_{a',a}} a')} \underline{x}^{c_{a',a}} a' & \text{if } \underline{x}^{b_{a,a'}}(e) \in (\mathbb{K}\langle \underline{x} \rangle^E)^{\mathrm{irr}}_{A^E,<^E}, \\ 0 & \text{otherwise,} \end{cases}$$
   where $b_{a,a'}, c_{a,a'} \in \mathbb{N}^n$ are given by $(b_{a,a'})_i := \max\{\mathrm{le}(a)_i, \mathrm{le}(a')_i\}$ and $(c_{a,a'})_i := (b_{a,a'})_i - \mathrm{le}(a)_i$ for $1 \leq i \leq n$. If $\mathrm{lcomp}(a) \neq \mathrm{lcomp}(a')$, we set $\mathrm{spoly}(a, a') := 0$.

3. The *s-polynomial* of a and $p \in I_e$ is defined by
   $$\mathrm{spoly}(a, p) := \begin{cases} \underline{x}^{c_{a,p}} a & \text{if } e = \mathrm{lcomp}(a), \\ 0 & \text{otherwise,} \end{cases}$$
   where $c_{a,p} \in \mathbb{N}^n$ is given by $(c_{a,p})_i := \max\{\mathrm{le}(a)_i, \mathrm{le}(p)_i\} - \mathrm{le}(a)_i$ for $1 \leq i \leq n$.

Remark 1.25. We keep the notation of Definition 1.24. Assume that $a, a' \in A^E$ satisfy $e := \mathrm{lcomp}(a) = \mathrm{lcomp}(a')$. Then

$$\mathrm{ele}(\mathrm{spoly}(a, a')) <^E (b_{a,a'}, e) = \mathrm{ele}(\underline{x}^{c_{a,a'}} a) = \mathrm{ele}(\underline{x}^{c_{a',a}} a').$$

Similarly, we have for $p \in I_e$

$$\mathrm{ele}(\mathrm{spoly}(a, p)) <^E c_{a,p} + \mathrm{ele}(a).$$

The following algorithm clearly computes a normal form and terminates, hence showing the existence of normal forms:

Algorithm 1.26 Given a PBW-reduction-algebra A, a finite set $G \subseteq A^E$, a well-ordering $\prec^E$ on A^E and $a \in A^E$, this algorithm computes a normal form of a with respect to G and $\prec^E$.

Input: A PBW-reduction-algebra A, a finite set E, $(A^E, \prec^E) = (\mathbb{K}\langle \underline{x} \rangle, S_e, I_e, \prec_e)_{e \in E}$, $G \subseteq A^E$ finite and $a \in A^E$.

Output: A normal form $b \in A^E$ of a with respect to G.

1: **while** $a \neq 0$ and $\tilde{G} := \{g \in G \mid \mathrm{ele}_{\prec^E}(a) \in L_{\prec^E}(\{g\})\} \neq \emptyset$ **do**
2: Choose $g \in \tilde{G}$.
3: Set $a := \mathrm{lc}_{\prec^E}(a) \cdot \mathrm{spoly}(a, g)$.
4: **return** a.

The above algorithm can be modified to return a reduced normal form using the same method as in the commutative setting (see e.g. (Greuel and Pfister, 2008, Algorithm 1.6.11)).

Remark 1.27. Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ a well-ordering on A^E and $M \subseteq A^E$ an A-submodule. If G is a Gröbner basis of M, then clearly $m \in A^E$ is an element of M if and only if some / every normal form of m with respect to G is 0.

Our algorithm for computing Gröbner bases is based on a variant of the Buchberger criterion for polynomial rings that takes into account the additional relations:

Proposition 1.28. *[Buchberger criterion for PBW-reduction-algebras] Let A be a PBW-reduction-algebra, E a finite set,* $(A^E, \prec^E) = (\mathbb{K}\langle \underline{x} \rangle, S_e, I_e, \prec_e)_{e \in E}$ *and* $G \subseteq A^E$ *a finite set. Then G is a (left) Gröbner basis (with respect to* $\prec^E$*) of the A-module* ${}_A\langle G \rangle$ *if and only if*

1. *for all* $g, g' \in G$ *some / any normal form of* $\mathrm{spoly}(g, g')$ *with respect to G is* 0 *and*

2. *for all* $g \in G$ *and* $p \in I_{\mathrm{lcomp}(g)}$ *some / any normal form of* $\mathrm{spoly}(a, g)$ *with respect to G is* 0.

For the proof we adapt a standard proof of the commutative Buchberger criterion to our setting. It relies on the following lemma, whose proof from the commutative setting carries over word by word:

Lemma 1.29. *Let A be a PBW-reduction-algebra, E a finite set,* $(A^E, \prec^E) = (\mathbb{K}\langle \underline{x} \rangle, S_e, I_e, \prec_e)_{e \in E}$*. Let* $G \subseteq A^E \setminus \{0\}$ *be a finite set whose elements have the same leading monomial. Let* $m = \sum_{g \in G} a_g g$ *with* $a \in \mathbb{K}^G$ *be such that* $\mathrm{lm}(m) \prec^E \mathrm{lm}(g)$ *for* $g \in G$*. Then there exists* $d \in \mathbb{K}^{G \times G}$ *such that* $m = \sum_{(g,g') \in G \times G} d_{(g,g')} \,\mathrm{spoly}(g, g')$.

The following remark lists some facts that are used throughout our proof of Proposition 1.28:

Remark 1.30. Let A be a PBW-reduction-algebra, E a finite set, $\prec_o^E$ an ordering on $(A^E, \prec^E) = (\mathbb{K}\langle \underline{x} \rangle, S_e, I_e, \prec_e)_{e \in E}$. Define for $l \in \mathbb{N}$, $1 \leq i_1, \ldots, i_l \leq n$ the vector $\alpha := \sum_{1 \leq j \leq l} e_{i_j} \in \mathbb{N}^n$ and let $e \in E$.

1. We have $\underline{x}^\alpha(e) \preceq_o^E x_{i_1} \cdots x_{i_l}(e)$.

2. Independently of the choice of $\prec_o^E$, we can find $r_{i_1,\ldots,i_l} \in \mathbb{K}[\underline{x}]$ and $f_{i_1,\ldots,i_l} \in \mathbb{K}^*$ with $\mathrm{ele}_{\prec_o^E}(r_{i_1,\ldots,i_l}(e)) \prec_o^E (\alpha, e)$ such that

$$x_{i_1} \cdots x_{i_l} - f_{i_1,\ldots,i_l} \underline{x}^\alpha - r_{i_1,\ldots,i_l} \in {}_{\mathbb{K}\langle \underline{x} \rangle}\langle S_e \rangle_{\mathbb{K}\langle \underline{x} \rangle}$$

and hence

$$\overline{x_{i_1}\cdots x_{i_l}(e)} = \overline{f_{i_1,\dots,i_l}\underline{x}^{\alpha}(e) + r_{i_1,\dots,i_l}(e)} \in A^E.$$

In particular, for any permutation σ of the set $\{1,\dots,l\}$

$$\overline{\frac{1}{f_{i_1,\dots,i_l}} x_{i_1}\cdots x_{i_l}(e) - \frac{1}{f_{i_{\sigma(1)},\dots,i_{\sigma(l)}}} x_{i_{\sigma(1)}}\cdots x_{i_{\sigma(l)}}(e)} = \bar{t}$$

for some $t \in \mathbb{K}[\underline{x}]^E$ with $\mathrm{lm}_{\prec_o^E}(t) \prec_o^E (\alpha, e)$. Suppose now $\prec_o^E = \prec^E$ is fixed. If $\underline{x}^{\alpha}(e) \in (\mathbb{K}\langle\underline{x}\rangle^E)^{\mathrm{irr}}_{A^E,\prec^E}$ then $f_{i_1,\dots,i_l}$ and $r_{i_1,\dots,i_l}$ can be additionally chosen such that

$$\rho_{A^E,\prec^E}(x_{i_1}\cdots x_{i_l}(e)) = f_{i_1,\dots,i_l}\underline{x}^{\alpha}(e) + r_{i_1,\dots,i_l}(e).$$

Otherwise $\mathrm{ele}_{\prec^E}(\rho_{A^E,\prec^E}(x_{i_1}\cdots x_{i_l}(e))) \prec^E (\alpha, e)$.

3. Let $a \in A$ and $g \in A^E$. Then $\mathrm{ele}_{\prec^E}(ag) \preceq^E \mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(a) + \mathrm{ele}_{\prec^E}(g)$ with equality if and only if the monomial with extended leading exponent $\mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(a) + \mathrm{ele}_{\prec^E}(g)$ is irreducible.

Proof of Proposition 1.28. By Remark 1.27 it is clear that if G is a Gröbner basis, then every normal form stated in the criterion is 0. Conversely, consider $0 \neq m \in {}_A\langle G\rangle$ and choose $h \in A^G$ such that

$$m = \sum_{g\in G} h_g g \tag{4}$$

satisfying additionally that

$$(\alpha, e) := \max_{\prec^E}\{\mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(h_g) + \mathrm{ele}_{\prec^E}(g) \mid g \in G\}$$

is minimal with respect to $\prec^E$. If $(\alpha, e) \preceq^E \mathrm{ele}_{\prec^E}(m)$ then Equation (4) is a standard representation and we are finished. Otherwise, set

$$G' := \{g \in G \mid \mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(h_g) + \mathrm{ele}_{\prec^E}(g) = (\alpha, e)\}$$

and write

$$m = l + \sum_{g'\in G'} \mathrm{tail}_{\prec_e^E}(h_{g'})g' + \sum_{g\in G\setminus G'} h_g g \quad \text{with} \quad l = \sum_{g'\in G'} \mathrm{lt}_{\prec_e^E}(h_{g'})g'. \tag{5}$$

By Remark 1.30.3 and by choice of G', we have for $g' \in G'$

$$\begin{aligned} \mathrm{ele}_{\prec^E}(\mathrm{tail}_{\prec_e^E}(h_{g'})g') &\preceq^E \mathrm{le}_{\prec_e^E}(\mathrm{tail}_{\prec_e^E}(h_{g'})) + \mathrm{ele}_{\prec^E}(g') \\ &\prec^E \mathrm{le}_{\prec_e^E}(h_{g'}) + \mathrm{ele}_{\prec^E}(g') = (\alpha, e), \end{aligned} \tag{6}$$

and for $g \in G \setminus G'$

$$\mathrm{ele}_{\prec^E}(h_g g) \preceq^E \mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(h_g) + \mathrm{ele}_{\prec^E}(g) \prec^E (\alpha, e). \tag{7}$$

Hence the leading monomial of l is strictly smaller than $\underline{x}^{\alpha}(e)$. We distinguish two cases: If $\underline{x}^{\alpha}(e) \in (\mathbb{K}\langle\underline{x}\rangle^E)^{\mathrm{irr}}_{A^E,\prec^E}$ then all summands of l have leading monomial $\underline{x}^{\alpha}(e)$ according to Remark 1.30.3. So Lemma 1.29 yields coefficients $d \in \mathbb{K}^{G'\times G'}$ to write

$$l = \sum_{(g,g')\in G'\times G'} d_{(g,g')} s_{(g,g')}, \tag{8}$$

as a linear combination of s-polynomials

$$s_{(g,g')} = \mathrm{spoly}(\mathrm{lm}_{\prec_e^E}(h_g)g, \mathrm{lm}_{\prec_e^E}(h_{g'})g') \tag{9}$$
$$= \frac{1}{\mathrm{lc}_{\prec^E}(\mathrm{lm}_{\prec_e^E}(h_g)g)} \mathrm{lm}_{\prec_e^E}(h_g)g - \frac{1}{\mathrm{lc}_{\prec^E}(\mathrm{lm}_{\prec_e^E}(h_{g'})g')} \mathrm{lm}_{\prec_e^E}(h_{g'})g'.$$

By definition of $c_{g,g'}$ and $c_{g',g}$ (see Definition 1.24.2) there exists $\beta_{(g,g')} \in \mathbb{N}^n$ such that $c_{g,g'} + \beta_{(g,g')} = \mathrm{le}_{\prec_e^E}(h_g)$ and $c_{g',g} + \beta_{(g,g')} = \mathrm{le}_{\prec_e^E}(h_{g'})$. Applying Remark 1.30.2, we obtain

$$s_{(g,g')} = (d_g \underline{x}^{\beta_{(g,g')}} \underline{x}^{c_{g,g'}} + r^{(g,g')})g - (d_{g'} \underline{x}^{\beta_{(g,g')}} \underline{x}^{c_{g',g}} + r^{(g',g)})g'$$
$$= \underline{x}^{\beta_{(g,g')}} \left(d_g \underline{x}^{c_{g,g'}} g - d_{g'} \underline{x}^{c_{g',g}} g' \right) + r^{(g,g')} g + r^{(g',g)} g'$$

for suitably chosen $d_g, d_{g'} \in \mathbb{K}^*$ and $r^{(g,g')}, r^{(g',g)} \in A$ with

$$\mathrm{lm}_{\prec_e^E}(r^{(g,g')}) \prec_e^E \mathrm{lm}_{\prec_e^E}(h_g) \text{ and } \mathrm{lm}_{\prec_e^E}(r^{(g',g)}) \prec_e^E \mathrm{lm}_{\prec_e^E}(h_{g'}).$$

Adding $\mathrm{ele}_{\prec^E}(g)$ and $\mathrm{ele}_{\prec^E}(g')$ respectively, we obtain

$$\mathrm{lm}_{\prec_e^E}(r^{(g,g')}) + \mathrm{ele}_{\prec^E}(g),\ \mathrm{lm}_{\prec_e^E}(r^{(g',g)}) + \mathrm{ele}_{\prec^E}(g') \prec_e^E (\alpha, e). \tag{10}$$

As $\underline{x}^\alpha(e)$ is irreducible and $c_{g,g'} + \beta_{(g,g')} + \mathrm{lm}_{\prec^E}(g) = (\alpha, e) = c_{g',g} + \beta_{(g,g')} + \mathrm{lm}_{\prec^E}(g')$, the monomial with extended leading coefficient $c_{g,g'} + \mathrm{lm}_{\prec^E}(g) = c_{g',g} + \mathrm{lm}_{\prec^E}(g')$ is also irreducible. By Equation (9) and Remark 1.25 $\mathrm{lm}_{\prec^E}(s_{(g,g')}) \prec \underline{x}^\alpha(e)$. Using Equation (10) it follows that $\mathrm{lt}_{\prec^E}(d_g \underline{x}^{c_{g,g'}} g) = \mathrm{lt}_{\prec^E}(d_{g'} \underline{x}^{c_{g',g}} g')$. By definition of $\mathrm{spoly}(g, g')$ it means that

$$d_g \underline{x}^{c_{g,g'}} g - d_{g'} \underline{x}^{c_{g',g}} g' = f_{(g,g')} \,\mathrm{spoly}(g, g')$$

for some $f_{(g,g')} \in \mathbb{K}^*$. Substituting into Equation (9) yields

$$s_{(g,g')} = f_{(g,g')} \underline{x}^{\beta_{(g,g')}} \,\mathrm{spoly}(g, g') + r^{(g,g')} g + r'^{(g,g')} g' \tag{11}$$

and

$$\beta_{(g,g')} + \mathrm{ele}_{\prec^E}(\mathrm{spoly}(g, g')) \prec^E (\alpha, e). \tag{12}$$

By hypothesis we find an element $k^{(g,g')} \in A^G$ satisfying

$$\mathrm{spoly}(g, g') = \sum_{g'' \in G} k_{g''}^{(g,g')} g'' \tag{13}$$

and $\mathrm{le}_{\prec^E_{\mathrm{lcomp}(g'')}}(k_{g''}^{(g,g')}) + \mathrm{ele}_{\prec^E}(g'') \preceq^E \mathrm{ele}_{\prec^E}(\mathrm{spoly}(g, g'))$. This yields together with Remark 1.30.3 and Equation (12) the estimate

$$\mathrm{le}_{\prec^E_{\mathrm{lcomp}(g'')}}(\underline{x}^{\beta(g,g')} k_{g''}^{(g,g')}) + \mathrm{ele}_{\prec^E}(g'') \preceq^E \beta_{(g,g')} + \mathrm{le}_{\prec^E_{\mathrm{lcomp}(g'')}}(k_{g''}^{(g,g')}) + \mathrm{ele}_{\prec^E}(g'') \tag{14}$$
$$\preceq^E \beta_{(g,g')} + \mathrm{ele}_{\prec^E}(\mathrm{spoly}(g, g'))$$
$$\prec^E (\alpha, e).$$

Combining Equations (8), (11) and (13) we obtain

$$l = \sum_{(g,g') \in G' \times G'} d_{(g,g')} \left(f_{(g,g')} \sum_{g'' \in G} \underline{x}^{\beta(g,g')} k_{g''}^{(g,g')} g'' + r^{(g,g')} g + r^{(g',g)} g' \right)$$

and substituting into Equation (5) contradicts the minimality of (α, e) by Equations (6), (7), (10) and (14).

In the other case, $\underline{x}^{\alpha}(e)$ is reducible, say $\alpha = \beta + \operatorname{lm}_{\prec_e^E}(p)$ for some $p \in I_e$ and $\beta \in \mathbb{N}^n$. Then there exists by definition of $\operatorname{spoly}(g, p)$ for $g \in G'$ a vector $\gamma_g \in \mathbb{N}^n$ such that

$$\operatorname{le}_{\prec_e^E}(h_g) + \operatorname{ele}_{\prec^E}(g) = (\alpha, e) = \gamma_g + c_{g,p} + \operatorname{ele}_{\prec_E}(g)$$

(see Definition 1.24.3 for the definition $c_{g,p}$). Therefore there is $q_g \in \mathbb{K}^*$

$$\operatorname{lm}_{\prec_e^E}(h_g)g = (q_g \underline{x}^{\gamma_g} \cdot \underline{x}^{c_{g,p}} + t_g)g = q_g \underline{x}^{\gamma_g} \cdot \operatorname{spoly}(g, p) + t_g g$$

with $t_g \in A$ such that $\operatorname{le}_{\prec_e^E}(t_g) \prec_e^E \operatorname{le}_{\prec_e^E}(h_g)$ by Remark 1.30.2. Using that

$$\gamma_g + \operatorname{ele}_{\prec^E}(\operatorname{spoly}(p, g)) \prec^E \gamma_g + c_{g,p} + \operatorname{ele}_{\prec_E}(g) = (\alpha, e)$$

by Remark 1.25 and that $\operatorname{spoly}(g, p)$ has a vanishing normal form with respect to G, we may argue as in the first case. This finishes our proof. □

The above lemma yields the following algorithm for computing Gröbner bases:

Algorithm 1.31 Given a PBW-reduction-algebra A, a well-ordering $\prec^E$ and a finite set $G \subseteq A^E$, this algorithm computes a Gröbner basis of the module ${}_A\langle G \rangle$ with respect to $\prec^E$.

Input: A PBW-reduction-algebra A, a finite set E, $(A^E, \prec^E) = (\mathbb{K}\langle \underline{x} \rangle, S_e, I_e, \prec_e)_{e \in E}$ and $G \subseteq A^E$ finite.

Output: A finite set $H \subseteq A^E$ such that H is a Gröbner basis of ${}_A\langle G \rangle$ with respect to $\prec^E$.

Initialize $H := G \setminus \{0\} := \{g_1, \ldots, g_s\}$.
Set $T := \{(g_i, g_j) \mid 1 \le i < j \le s\} \cup \{(g, p) \mid g \in H, p \in I_{\operatorname{lcomp}(g)}\}$.
while $T \neq \emptyset$ **do**
  Choose $(t_1, t_2) \in T$ and delete it from T.
  Compute a normal form r of $\operatorname{spoly}(t_1, t_2)$ with respect to H and $\prec^E$ by applying Algorithm 1.26.
  if $r \neq 0$ **then**
    Set $T := T \cup \{(r, h) \mid h \in H\} \cup \{(r, p) \mid p \in I_{\operatorname{lcomp}(r)}\}$ and $H := H \cup \{r\}$.
return H.

Lemma 1.32. *Algorithm 1.31 is correct and terminates.*

Proof. Correctness follows immediately from Proposition 1.28. The $\mathrm{L}(H)$ form an increasing sequence of $\mathbb{N}^n$-stable subsets of $\mathbb{N}^n \times E$. By definition of a normal form it stabilizes exactly if H does. Elements of $\mathbb{N}^n \times E$ identify with monomials in $\mathbb{K}[\underline{x}]^E$. The latter is Noetherian and termination follows. □

As in the commutative setting, the above algorithm can be modified to compute a reduced Gröbner basis. An algorithm for computing left generators of a two-sided submodule of a free A-module carries over immediately from the setting of PBW-algebras (see e.g. (Bueso et al., 2003, Algorithm 6)). In our setting termination is a consequence of Proposition 1.19. Together with Lemma 1.32 this yields:

Proposition 1.33. *Let A be a PBW-reduction-algebra, E a finite set, $(A^E, \prec^E) = (\mathbb{K}\langle \underline{x}\rangle, S_e, I_e, \prec_e)_{e\in E}$ and $G \subseteq A^E$ a finite subset. Then (reduced) Gröbner bases of the left A-modules ${}_A\langle G\rangle$ and ${}_A\langle G\rangle_A$ with respect to $\prec^E$ are computable.*

Definition 1.34. We call $\mathbb{K}\langle \underline{x}\rangle \,/\, \langle I' \cup S\,\rangle = (\mathbb{K}\langle \underline{x}\rangle, S, I', \prec)$ a *factor PBW-reduction-algebra* of $A = (\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)$ if $I \subseteq I'$.

The following result explains how we consider factor algebras of PBW-reduction-algebras as PBW-reduction-algebras.

Corollary 1.35. *Let $A = (\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)$ be a PBW-reduction-algebra and $M \subseteq A$ a two-sided A-ideal. Then A/M is canonically isomorphic to the PBW-reduction-algebra*

$$\mathbb{K}\langle \underline{x}\rangle \,/\, \langle S \cup I \cup \tau_{A,\prec}(G)\rangle = (\mathbb{K}\langle \underline{x}\rangle, S, I \cup \tau_{A,\prec}(G), \prec),$$

where G is a left Gröbner basis of M with respect to $\prec$.

Proof. Clearly the map $\mathbb{K}\langle \underline{x}\rangle \to A$, $t \mapsto \bar{t}$ induces the claimed isomorphism. For the second claim it is by Remark 1.9 enough to show that

$$\mathrm{L}_\prec(I \cup \tau_{(A,\prec)}(G)) \supseteq \mathrm{L}_\prec({}_{\mathbb{K}\langle \underline{x}\rangle}\langle S \cup I \cup \tau_{A,\prec}(G)\rangle_{\mathbb{K}\langle \underline{x}\rangle} \cap \mathbb{K}[\underline{x}]).$$

So consider $0 \neq t \in {}_{\mathbb{K}\langle \underline{x}\rangle}\langle S \cup I \cup \tau_{A,\prec}(G)\rangle_{\mathbb{K}\langle \underline{x}\rangle} \cap \mathbb{K}[\underline{x}]$. If $\mathrm{le}(t) \in \mathrm{L}_\prec(I)$, we are finished. Otherwise we have according to Equation (1) that $\mathrm{lm}(t)$ is irreducible with respect to $({}_{\mathbb{K}\langle \underline{x}\rangle}\langle S \cup I\rangle_{\mathbb{K}\langle \underline{x}\rangle}, \prec)$ and hence $\mathrm{lm}(t) = \mathrm{lm}(\rho_{A,\prec}(t)) = \mathrm{lm}(\bar{t})$. By construction $\bar{t} \in M$ and for suitable coefficients $a \in A^G$ there is a standard representation with respect to G

$$\bar{t} = \sum_{g\in G} a_g g \text{ and } \mathrm{le}(a_g) + \mathrm{le}(g) \leq \mathrm{le}(\bar{t}) = \mathrm{le}(t) \text{ for all } g \in G$$

with equality for some $g' \in G$. With $\mathrm{le}(g') = \mathrm{le}(\tau_{A,\prec}(g'))$ the claim follows. □

Definition 1.36. Let A be a ring, E a finite set and $H_1, \dots, H_s \subseteq A^E$ finite subsets. The A-module

$$\mathrm{syz}_A(H_1, \dots, H_s) := \{(a_1, \dots, a_s) \in A^{H_1} \oplus \cdots \oplus A^{H_s} \mid \sum_{1\leq i\leq s} \sum_{h_i\in H_i} (a_i)_{h_i} h_i = 0\}$$

is called the *syzygy-module* of $H_1, \dots, H_s$ (in $A^{H_1} \oplus \cdots \oplus A^{H_s}$). Similarly, for $h_1, \dots, h_t \in A^E$ we define the *syzygy-module* $\mathrm{syz}_A(h_1, \dots, h_t) := \mathrm{syz}_A(\{h_1\}, \dots, \{h_t\})$.

Syzygies over PBW-reduction-algebras can be computed as in the commutative case:

Lemma 1.37. *Let $A = (\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)$ be a PBW-reduction-algebra, E a finite set and $H \subseteq A^E$ finite. Let G be a Gröbner basis of ${}_A\langle\{h + (h) \mid h \in H\}\rangle \subseteq A^{E\sqcup H}$ with respect to $(<, \prec^{E\sqcup H})$, where $<$ is a total ordering on $E \sqcup H$ with $h < e$ for $e \in E$ and $h \in H$. Then*

$$\mathrm{syz}_A(H) = {}_A\Big\langle \pi_H(G \cap A^H)\Big\rangle.$$

Remark 1.38. Given a PBW-reduction algebra $A = (\mathbb{K}\langle \underline{x}\rangle, S, I, \prec)$ and a finite set E the following Gröbner basics can be performed as in the commutative setting:

1. We can decide for submodules of A^E whether one is included in the other using normal form computations with respect to any ordering $\prec^E$ with $\prec^E_e = \prec$ for all $e \in E$ (see Remark 1.21(1)).

2. A non-commutative variant of (Greuel and Pfister, 2008, Section 2.8.3) allows to compute intersections of submodules of A^E,

In the next section, we explain how to compute Gröbner bases with respect to non-well-orderings.

2. Weight filtrations

The subject of investigation in this section are filtrations of type $F^{\mathbf{u}}_{\bullet}A$ induced by a so-called weight vector $\mathbf{u}$ on the PBW-reduction-algebra A. These filtrations have been studied theoretically and algorithmically for nonnegative weight vectors on PBW-algebras in Bueso et al. (2003). Combining the methods of Bueso et al. (2003) and Oaku and Takayama (2001), we develop a Gröbner basis algorithm for computing $F^{\mathbf{u}}_{\bullet}A$ for general weight vectors $\mathbf{u}$.

2.1. Weight filtrations on PBW-reduction-algebras

In this subsection let $A = (\mathbb{K}\langle \underline{x} \rangle, S, I, \prec)$ be a PBW-reduction-algebra unless stated otherwise.

Definition 2.1. Let $\mathbf{u} \in \mathbb{Z}^n$, E a finite set, $L \subseteq A^E$ an A-submodule and $\mathbf{s} \in \mathbb{Z}^E$.

1. Assigning weight $\mathbf{u}_i$ to x_i and weight $\mathbf{s}_e$ to (e) defines a $\mathbb{Z}$-grading on $\mathbb{K}\langle \underline{x} \rangle^E$ with lth graded piece $(\mathbb{K}\langle \underline{x} \rangle^E)^{\mathbf{u}[\mathbf{s}]}_l = \bigoplus_{e \in E} \mathbb{K}\langle \underline{x} \rangle^{\mathbf{u}}_{l-\mathbf{s}_e}(e)$ with
$$\mathbb{K}\langle \underline{x} \rangle^{\mathbf{u}}_l := {}_{\mathbb{K}}\Big\langle \{x_{i_1} \cdots x_{i_k} \mid k \in \mathbb{N}, 1 \le i_1, \dots, i_k \le n, \sum\nolimits_{1 \le j \le k} \mathbf{u}_{i_j} = l\} \Big\rangle.$$
So every $0 \neq r \in \mathbb{K}\langle \underline{x} \rangle^E$ can be uniquely written as $r = \sum_{s_1 \le i \le s_2} r_i$ with $r_i \in (\mathbb{K}\langle \underline{x} \rangle^E)^{\mathbf{u}[\mathbf{s}]}_i$ and $r_{s_1}, r_{s_2} \neq 0$. We call s_2 the $\mathbf{u}[\mathbf{s}]$-*degree* of r and write $\deg_{\mathbf{u}[\mathbf{s}]}(r) = s_2$. We set $\deg_{\mathbf{u}[\mathbf{s}]}(0) := -\infty$. If $s_1 = s_2$, we say that r is $\mathbf{u}[\mathbf{s}]$-*homogeneous*. We define the $\mathbf{u}[\mathbf{s}]$-*leading terms* of r by $\mathrm{lt}_{\mathbf{u}[\mathbf{s}]}(r) := r_{s_2}$. The elements $r_{s_1}, \dots, r_{s_2}$ are called the $\mathbf{u}[\mathbf{s}]$-*homogeneous parts* of r.

2. The associated filtration on $F^{\mathbf{u}}[\mathbf{s}]_{\bullet}$ on $\mathbb{K}\langle \underline{x} \rangle^E$ is defined by
$$F^{\mathbf{u}}[\mathbf{s}]_k \, \mathbb{K}\langle \underline{x} \rangle^E := \{r \in \mathbb{K}\langle \underline{x} \rangle^E \mid \deg_{\mathbf{u}[\mathbf{s}]}(r) \le k\}$$
for $k \in \mathbb{Z}$. It induces a quotient filtration $F^{\mathbf{u}}[\mathbf{s}]_{\bullet}A^E := \bigoplus_{e \in E} F^{\mathbf{u}}_{\bullet - \mathbf{s}_e}A(e)$, where $F^{\mathbf{u}}_{\bullet}A := (F^{\mathbf{u}}_{\bullet}\, \mathbb{K}\langle \underline{x} \rangle + \langle I \cup S \rangle)/\langle I \cup S \rangle$. We define the $\mathbf{u}[\mathbf{s}]$-degree for $a \in A^E$ by
$$\deg_{\mathbf{u}[\mathbf{s}]}(a) := \deg_{F^{\mathbf{u}}[\mathbf{s}]}(a) := \inf\{k \in \mathbb{Z} \mid a \in F^{\mathbf{u}}[\mathbf{s}]_k A^E\}$$
and extend it to subsets T of A^E or $\mathbb{K}\langle \underline{x} \rangle^E$ is by setting $\deg_{\mathbf{u}[\mathbf{s}]}(T) := \max\{\deg_{\mathbf{u}[\mathbf{s}]}(t) \mid t \in T\}$.

3. Using the induced filtrations $F^{\mathbf{u}}[\mathbf{s}]_{\bullet}L := F^{\mathbf{u}}[\mathbf{s}]_{\bullet}A^E \cap L$ and $F^{\mathbf{u}}[\mathbf{s}]_{\bullet}(A^E/L) := (F^{\mathbf{u}}[\mathbf{s}]_{\bullet}A^E + L)/L$, we sets the $\mathbf{u}[\mathbf{s}]$-degree of elements and subsets of L and A^E/L as in Part 2.

4. We call $\mathrm{Gr}^{\mathbf{u}}A := \bigoplus_{k \in \mathbb{Z}} F^{\mathbf{u}}_k A / F^{\mathbf{u}}_{k-1}A$ the $\mathbf{u}$-*graded algebra associated with* A and $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]}L := \bigoplus_{k \in \mathbb{Z}} F^{\mathbf{u}}[\mathbf{s}]_k L / F^{\mathbf{u}}[\mathbf{s}]_{k-1} L$ $\mathbf{u}$-*graded module associated with* L.

5. We say that $\mathbf{u}$ is a *weight vector* on A if $\deg_{\mathbf{u}}(d_{ij}) \leq \deg_{\mathbf{u}}(x_i x_j)$ for all $1 \leq i < j \leq n$, where $S = \{x_j x_i - c_{ij} x_i x_j - d_{ij} \mid 1 \leq i < j \leq n\}$. In this case we call $F^{\mathbf{u}}_{\bullet} A$ the *weight filtration* associated to $\mathbf{u}$ on A or the $\mathbf{u}$-*weight filtration* on A. If $A \cong \mathrm{Gr}^{\mathbf{u}} A$, then we say that A is $\mathbf{u}$-*graded* and speak of $\mathbf{u}$-*homogeneous* elements of A. More generally, if A is $\mathbf{u}$-graded, E a finite set and the shift vector $\mathbf{s} \in \mathbb{Z}^E$ assigns degree $\mathbf{s}_e$ to (e), then we call a homogeneous element of A^E also $\mathbf{u}[\mathbf{s}]$-homogeneous.

We often suppress $\mathbf{s}$ in the above notations if it is the zero vector.

Lemma 2.2. *Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A, E a finite set, $\mathbf{s} \in \mathbb{Z}^E$ and $L \subseteq A^E$ an A-submodule. Then we have for all $a, a' \in A$*

$$\deg_{\mathbf{u}}(a \cdot a') \leq \deg_{\mathbf{u}}(a) + \deg_{\mathbf{u}}(a').$$

In particular, $F^{\mathbf{u}}_{\bullet} A = {}_{\mathbb{K}}\left\langle \{\overline{\underline{x}^{\alpha}} \mid \langle \mathbf{u}, \alpha \rangle \leq \bullet\} \right\rangle$ is a filtered $\mathbb{K}$-algebra and $F^{\mathbf{u}}[\mathbf{s}]_{\bullet} A^E$, $F^{\mathbf{u}}[\mathbf{s}]_{\bullet} L$ and $F^{\mathbf{u}}[\mathbf{s}]_{\bullet}(A^E/L)$ are filtered $F^{\mathbf{u}}_{\bullet} A$-modules.

Definition 2.3. Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A, E a finite set, $<^E$ an ordering on A^E and $\mathbf{s} \in \mathbb{Z}^E$ a shift vector. We define the ordering $<^E_{\mathbf{u}[\mathbf{s}]}$ on $\mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle^E)$ by

$$\begin{aligned} \underline{x}^{\alpha}(e) <^E_{\mathbf{u}[\mathbf{s}]} \underline{x}^{\alpha'}(e') \text{ if and only if } & \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha}(e)) < \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha'}(e')) \\ \text{or } & \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha}(e)) = \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha'}(e')) \text{ and } \underline{x}^{\alpha}(e) <^E \underline{x}^{\alpha'}(e') \end{aligned}$$

for $\alpha, \alpha' \in \mathbb{N}^n$ and $e, e' \in E$. If $\mathbf{s}$ is the zero vector, we also write $<^E_{\mathbf{u}}$. We sometimes use the notation $<^E_{\mathbf{u}[\mathbf{s}]}$ without explicitly defining an ordering $<^E$ on A^E.

Lemma 2.2 implies that $F^{\mathbf{u}}_0 A$ is a $\mathbb{K}$-subalgebra of A if $\mathbf{u}$ is a weight vector on A.

Lemma 2.4. *Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A.*

1. *The $\mathbb{K}$-subalgebra $F^{\mathbf{u}}_0 A$ of A is generated by residue classes of finitely many standard monomials. Moreover, such a generating set is computable.*

2. *The $\mathbb{K}$-subalgebra $F^{\mathbf{u}}_0 A$ is isomorphic to a PBW-reduction-algebra.*

3. *The $F^{\mathbf{u}}_0 A$-modules $F^{\mathbf{u}}_j A$ ($j \in \mathbb{Z}$) are generated by residue classes of finitely many standard monomials. Moreover, such generating sets are computable.*

Proof.

1. Taking exponents $\mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle) \cap F^{\mathbf{u}}_0 \mathbb{K}\langle \underline{x} \rangle$ identifies with

$$U_0 := \{\alpha \in \mathbb{R}^n \mid \langle \mathbf{u}, \alpha \rangle \leq 0\} \cap \mathbb{N}^n,$$

which is an intersection of a rational cone and the lattice $\mathbb{Z}^n$. Therefore U_0 is a positive affine monoid by Gordan's lemma (see e.g. (Bruns and Gubeladze, 2009, Lemma 2.9)), and has a computable minimal finite generating set (Koch, 2003, Proposition 3.4.6) (Bruns and Ichim, 2010), say $\alpha_1, \ldots, \alpha_s \in \mathbb{Z}^n$. This means that $U_0 = \sum_{1 \leq i \leq s} \mathbb{N} \cdot \alpha_i$, and if $\alpha_i = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in U_0$, then $\beta_1 = \alpha_i$ or $\beta_2 = \alpha_i$ for $1 \leq i \leq s$.

We claim that $F^{\mathbf{u}}_0 A = \mathbb{K}[\overline{\underline{x}^{\alpha_1}}, \ldots, \overline{\underline{x}^{\alpha_s}}]$. By Lemma 2.2, it suffices to show that $F^{\mathbf{u}}_0 \mathbb{K}\langle \underline{x} \rangle \cap \mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle)$ maps to $\mathbb{K}[\overline{\underline{x}^{\alpha_1}}, \ldots, \overline{\underline{x}^{\alpha_s}}]$. We proceed by induction on the well-ordering $<$. The base case is immediate since $1 = \min_{\prec}\{F_0^{\mathbf{u}}\,\mathbb{K}\langle \underline{x}\rangle \cap \mathrm{SMon}(\mathbb{K}\langle \underline{x}\rangle)\}$. Consider now $\underline{x}^{\alpha} \in F_0^{\mathbf{u}}\,\mathbb{K}\langle \underline{x}\rangle \cap \mathrm{SMon}(\mathbb{K}\langle \underline{x}\rangle)$. Then $\alpha \in U_0$, and hence $\alpha = \sum_{1\le i\le s} l_i\alpha_i$ for some $l \in \mathbb{N}^s$. By Remark 1.30(2) there exists $c \in \mathbb{K}^*$ and $a \in {}_{\mathbb{K}}\langle \mathrm{SMon}(\mathbb{K}\langle \underline{x}\rangle)\rangle$ with $\mathrm{lm}(a) \prec \underline{x}^{\alpha}$ such that

$$\overline{\underline{x}^{\alpha}} = \overline{\underline{x}^{\sum_{1\le i\le s} l_i\alpha_i}} = c\overline{(\underline{x}^{\alpha_1})^{l_1}\cdots(\underline{x}^{\alpha_s})^{l_s}} + \overline{a}.$$

The induction hypothesis applied to $\overline{a} \in F_0^{\mathbf{u}}A$ yields the claim.

2. We retain the notation of Part (1). Consider the surjective $\mathbb{K}$-algebra map

$$\pi : \mathbb{K}\langle \underline{y}\rangle := \mathbb{K}\langle y_1,\dots,y_s\rangle \to F_0^{\mathbf{u}}A,\ y_i \mapsto \overline{\underline{x}^{\alpha_i}}.$$

By Remark 1.30(2) with $\prec_o^E = \prec_{\mathbf{u}}$ there exist $f_{ij} \in \mathbb{K}^*$ and $g_{ij} \in \mathbb{K}[\underline{x}]$ with $\mathrm{le}_{\prec}(g_{ij}) \prec \alpha_i + \alpha_j$ and $\deg_{\mathbf{u}}(g_{ij}) \le \deg_{\mathbf{u}}(\underline{x}^{\alpha_i}\underline{x}^{\alpha_j}) \le 0$ such that

$$\underline{x}^{\alpha_j}\underline{x}^{\alpha_i} - f_{ij}\underline{x}^{\alpha_i}\underline{x}^{\alpha_j} - g_{ij} \in {}_{\mathbb{K}\langle \underline{x}\rangle}\langle S\rangle_{\mathbb{K}\langle \underline{x}\rangle} \subseteq {}_{\mathbb{K}\langle \underline{x}\rangle}\langle S, I\rangle_{\mathbb{K}\langle \underline{x}\rangle}$$

for $1 \le i < j \le s$. Then $g_{ij} \in F_0^{\mathbf{u}}A$ and Part (1) yield $g'_{ij}(y_1,\dots,y_s) \in \mathbb{K}[\underline{y}]$ such that $g'_{ij}(\overline{\underline{x}^{\alpha_1}},\dots,\overline{\underline{x}^{\alpha_s}}) = \overline{g_{ij}} \in A$. It follows

$$S_0 := \{y_jy_i - f_{ij}y_iy_j - g'_{ij} \mid 1 \le i < j \le s\} \subseteq \ker(\pi).$$

Define the well-ordering $\prec_0$ on $\mathrm{SMon}(\mathbb{K}\langle \underline{y}\rangle)$ by

$$\underline{y}^{\beta} \prec_0 \underline{y}^{\gamma} \text{ if and only if } \sum_{1\le k\le s}\beta_i\alpha_i \prec \sum_{1\le k\le s}\gamma_i\alpha_i$$
$$\text{or } \sum_{1\le k\le s}\beta_i\alpha_i = \sum_{1\le k\le s}\gamma_i\alpha_i \text{ and } \underline{y}^{\beta} \prec' \underline{y}^{\gamma},$$

where $\beta,\gamma \in \mathbb{N}^s$ and $\prec'$ is some well-ordering on $\mathrm{SMon}(\mathbb{K}\langle \underline{y}\rangle)$. By construction, $(S_0, \prec_0$ $)$ is a commutation system. We conclude that $\mathbb{K}\langle \underline{y}\rangle/\ker\pi$ is a PBW-reduction-algebra isomorphic to $F_0^{\mathbf{u}}A$ by Proposition 1.13.

3. We keep the notation of Part (1) and consider first the case $j < 0$. Let $\Delta_j := \{\alpha_i \mid \langle \mathbf{u}, \alpha_i\rangle \le j\}$ and $\Delta'_j := \{\alpha_i \mid j < \langle \mathbf{u}, \alpha_i\rangle < 0\}$. One easily checks that

$$U_j := \{\alpha \in \mathbb{N}^n \mid \langle \mathbf{u}, \alpha\rangle \le j\} = U_0 + V_j := \{\alpha + v \mid \alpha \in U_0, v \in V_j\}, \tag{15}$$

where $V_j := \Delta_j \cup (U_j \cap \{\sum_{\delta\in\Delta'} l_\delta\delta \mid l \in \mathbb{N}^{\Delta'}, |l| \le j\})$. We claim that

$$F_j^{\mathbf{u}}A = \sum_{v\in V_j} F_0^{\mathbf{u}}A\overline{\underline{x}^{v}}. \tag{16}$$

By Lemma 2.2 it suffices to show that $F_j^{\mathbf{u}}\,\mathbb{K}\langle \underline{x}\rangle \cap \mathrm{SMon}(\mathbb{K}\langle \underline{x}\rangle)$ maps to the right hand side. This set has a minimal element $\underline{x}^{\beta}$ with respect to $\prec$ and we must have $\beta \in V_j$. Proceeding by induction, the claim follows as in Part (1).

The case $j = 0$ being clear, we assume now $j > 0$. Arguing as in the proof of Part (1), we can compute a minimal finite set of generators Γ of $\{\alpha \in \mathbb{N}^n \mid \langle \mathbf{u}, \alpha\rangle \ge 0\}$. As above, we obtain

$$U_j := \{\alpha \in \mathbb{N}^n \mid \langle \mathbf{u}, \alpha\rangle \le j\} = U_0 + V_j,$$

where $V_j := (\{\sum_{\gamma\in\Gamma_j} l_\gamma\gamma \mid l \in \mathbb{N}^{\Gamma}, |l| \le j\} \cap (U_j \setminus U_0)) \cup \{0\}$ with $\Gamma_j := G \cap (U_j \setminus U_0)$. With this notation Equation (16) follows as in case $j < 0$. □

Notation 2.5. Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A.

1. With notation from the proof of Lemma 2.4.1 we denote the set $G_A^{\mathbf{u}} := \{\underline{x}^{\alpha_1}, \ldots, \underline{x}^{\alpha_s}\}$, whose residue classes generate $F_0^{\mathbf{u}}A$ as $\mathbb{K}$-algebra.

2. With notation from the proof of Lemma 2.4.3 we denote for $j \in \mathbb{Z}$ the set $P_j^{A,\mathbf{u}} := \{\overline{\underline{x}^v} \mid v \in V_j\}$, whose residue classes generate $F_j^{\mathbf{u}}A$ as $F_0^{\mathbf{u}}A$-module.

Remark 2.6. Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A. The proof of Lemma 2.4.1 and 3 is constructive. It gives a method to represent an element of $F_0^{\mathbf{u}}A$ as a $\mathbb{K}$-linear combination of products of elements in $G_A^{\mathbf{u}}$ and elements $F_j^{\mathbf{u}}A$ as $F_0^{\mathbf{u}}A$-linear combinations of elements in $P_j^{A,\mathbf{u}}$.

Remark 2.7.

1. Let $A = (\mathbb{K}\langle \underline{x}, \underline{y}\rangle, S, I, \prec)$ (with $\mathbb{K}\langle \underline{x}, \underline{y}\rangle := \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$) be an elementary PBW-reduction-algebra and $\mathbf{v} \in \mathbb{Z}^{n+m}$ be any weight vector on A. Note that $\mathbf{w} = ((0)_{1\le i\le n}, (1)_{1\le i\le m})$ is a weight vector on A. Then
$$F_k^{\mathbf{v}}A \cap F_l^{\mathbf{w}}A = \Big((F_k^{\mathbf{v}}\,\mathbb{K}\langle \underline{x}, \underline{y}\rangle \cap F_l^{\mathbf{w}}\,\mathbb{K}\langle \underline{x}, \underline{y}\rangle \cap \mathbb{K}[\underline{x}, \underline{y}]) + \langle I \cup S\rangle\Big) / \langle I \cup S\rangle$$
for all $k, l \in \mathbb{Z}$: Indeed let $a \in F_k^{\mathbf{v}}A \cap F_l^{\mathbf{w}}A$. By Lemma 2.2 there exist representatives $a^{\mathbf{w}} \in F_l^{\mathbf{w}}\,\mathbb{K}\langle \underline{x}, \underline{y}\rangle \cap \mathbb{K}[\underline{x}, \underline{y}]$ and $a^{\mathbf{v}} = \sum_{(\alpha,\beta)\in\mathbb{N}^{n+m}} a^{\mathbf{v}}_{(\alpha,\beta)}\underline{x}^\alpha\underline{y}^\beta \in F_k^{\mathbf{v}}\,\mathbb{K}\langle \underline{x}, \underline{y}\rangle$ of a. By Definition 1.16 $A = \bigoplus_{\beta\in\mathbb{N}^m}(\mathbb{K}[\underline{x}]/\langle I\rangle)\underline{y}^\beta$, and hence $\overline{\sum_{\alpha\in\mathbb{N}^n} a^{\mathbf{v}}_{(\alpha,\beta)}\underline{x}^\alpha} = 0 \in A$ for all $\beta \in \mathbb{N}^m$ with $|\beta| > l$. Thus $\sum_{(\alpha,\beta)\in\mathbb{N}^{n+m}, |\beta|\le l} a^{\mathbf{v}}_{(\alpha,\beta)}\underline{x}^\alpha\underline{y}^\beta \in F_k^{\mathbf{v}}\,\mathbb{K}\langle \underline{x}, \underline{y}\rangle \cap F_l^{\mathbf{w}}\,\mathbb{K}\langle \underline{x}, \underline{y}\rangle$ is also a representative of a.

2. Let $\mathbf{v}$ and $\mathbf{w} \in \mathbb{Z}^n$ be weight vectors on $A = (\mathbb{K}\langle \underline{\mathrm{x}}\rangle, S, I, \prec)$ such that
$$F_k^{\mathbf{v}}A \cap F_l^{\mathbf{w}}A = \Big((F_k^{\mathbf{v}}\,\mathbb{K}\langle \underline{\mathrm{x}}\rangle \cap F_l^{\mathbf{w}}\,\mathbb{K}\langle \underline{\mathrm{x}}\rangle \cap \mathbb{K}[\underline{\mathrm{x}}]) + \langle I \cup S\rangle\Big) / \langle I \cup S\rangle \tag{17}$$
for $k, l \in \mathbb{Z}$. By construction (see proof of Lemma 2.4(3))
$$F_\bullet^{\mathbf{w}}F_k^{\mathbf{v}}A = \sum_{p\in P_k^{A,\mathbf{v}}} (F^{\mathbf{w}}_{\bullet-\langle\beta_i^k,\mathbf{w}\rangle}F_0^{\mathbf{v}}A)\cdot\overline{p}.$$

Example 2.8. In the situation of Example 1.17.2, we have $T_X^V \cong F_0^{\mathbf{v}}T_X$, where $\mathbf{v}$ is the weight vector assigning weights -1 and 1 to x_n and y_m, respectively, and weight 0 otherwise. By this example and Remark 2.7 the weight vector $\mathbf{w} = ((0)_{1\le i\le n}, (1)_{1\le i\le m})$ on T_X induces the weight vector $\mathbf{w}_{\mathbf{v}} = ((0)_{1\le i\le n}, (1)_{1\le i\le m})$ on T_X^V. Moreover, we may assume
$$P_k^{T_X,\mathbf{v}} = \begin{cases}\{x_n^k\} & \text{if } k \le 0,\\ \{y_m^l \mid 0 \le l \le k\} & \text{otherwise}\end{cases}$$
and fix this choice.

2.2. *Weight filtrations on submodules of free modules*

In this subsection let $A = (\mathbb{K}\langle \underline{\mathrm{x}}\rangle, S, I, \prec)$ be a PBW-reduction-algebra unless otherwise specified and $\mathbf{u} \in \mathbb{Z}^n$ a weight vector on A. For a given set E, an A-submodule $M \subseteq A^E$ and a shift vector $\mathbf{s} \in \mathbb{Z}^E$ we show how to compute a finite set of generators M' of the filtration $F_\bullet^{\mathbf{u}}[\mathbf{s}]M$. By definition this means that for every $m \in M$ there are coefficients $a \in A^{M'}$ such that
$$m = \sum_{m'\in M'} a_{m'}m' \text{ and } \deg_{\mathbf{u}}(a_{m'}) + \deg_{\mathbf{u}[\mathbf{s}]}(m') \le \deg_{\mathbf{u}[\mathbf{s}]}(m) \text{ for all } m' \in M'.$$

Lemma 2.9. *Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A, E a finite set, $\mathbf{s} \in \mathbb{Z}^E$ a shift vector, $\prec^E$ an ordering and $M \subseteq A^E$ an A-submodule. If G is a Gröbner basis of M with respect to $\prec^E_{\mathbf{u}[\mathbf{s}]}$, then it generates $F^{\mathbf{u}}[\mathbf{s}]_\bullet M$ as $F^{\mathbf{u}}_\bullet A$-module.*

Proof. Let $m \in F^{\mathbf{u}}[\mathbf{s}]_k M$ for some $k \in \mathbb{Z}$. Choose a representative $m' \in F^{\mathbf{u}}[\mathbf{s}]_k\, \mathbb{K}\langle \underline{x}\rangle \cap \mathbb{K}[\underline{x}]$ of m using Lemma 2.2. By assumption there is $a \in {}_{\mathbb{K}}\langle \mathrm{SMon}(\mathbb{K}\langle \underline{x}\rangle)\rangle^G$ and $h \in (\mathbb{K}[\underline{x}]^E)^G$ with $\overline{h_g} = g$ satisfying

$$m = \sum_{g\in G} \overline{a_g} g \text{ and } \mathrm{le}(a_g) + \mathrm{ele}(h_g) \preceq^E_{\mathbf{u}[\mathbf{s}]} \mathrm{ele}(m')$$

implying $\deg_{\mathbf{u}}(\overline{a_g}) + \deg_{\mathbf{u}[\mathbf{s}]}(g) \le \deg_{\mathbf{u}}(a_g) + \deg_{\mathbf{u}[\mathbf{s}]}(h_g) \le \deg_{\mathbf{u}}(m') \le k$. This means that $m \in \sum_{g\in G} F^{\mathbf{u}}_{k-\deg_{\mathbf{u}[\mathbf{s}]}(g)} A \cdot g$. □

Note that if $\prec^E$ is a well-ordering, then $\prec^E_{\mathbf{u}[\mathbf{s}]}$ is a well-ordering if and only if $\mathbf{u} \in \mathbb{N}^n$. Since Gröbner bases with respect to well-orderings exist by Proposition 1.33, a finite set of generators of $F^{\mathbf{u}}_\bullet M$ exists in this case. If $\mathbf{u} \notin \mathbb{N}^n$, we can still compute Gröbner bases with respect to $\prec^E_{\mathbf{u}[\mathbf{s}]}$. To this end we homogenize A with respect to a weight vector $\mathbf{w}$:

Definition 2.10. Let $\mathbf{w} \in \mathbb{N}^n$ be a weight vector on A, E a finite set and $\mathbf{s} \in \mathbb{Z}^E$ a shift vector.

1. We define the $\mathbf{w}[\mathbf{s}]$-*homogenization* of $0 \ne p = \sum_{m\in \mathrm{Mon}(\mathbb{K}\langle \underline{x}\rangle^E)} p_m m \in \mathbb{K}\langle \underline{x}\rangle^E$ (with $p_m \in \mathbb{K}$) as
$$h_{\mathbf{w}[\mathbf{s}]}(p) := \sum_{m\in \mathrm{Mon}(\mathbb{K}\langle \underline{x}\rangle^E)} p_m h^{\deg_{\mathbf{w}[\mathbf{s}]}(p)-\deg_{\mathbf{w}[\mathbf{s}]}(m)} m \in \mathbb{K}\langle h, \underline{x}\rangle^E$$
and set $h_{\mathbf{w}[\mathbf{s}]}(0) = 0$. For $G \subseteq \mathbb{K}\langle \underline{x}\rangle^E$, we set $h_{\mathbf{w}[\mathbf{s}]}(G) := \{h_{\mathbf{w}[\mathbf{s}]}(g) \mid g \in G\}$. We suppress $\mathbf{s}$ if it is the zero vector.

2. The $\mathbf{w}$-*homogenized* algebra associated with A is the $(1, \mathbf{w})$-graded algebra
$$A^{\mathbf{w}} = \mathbb{K}\langle h, \underline{x}\rangle / \left\langle h_{\mathbf{w}}({}_{\mathbb{K}\langle \underline{x}\rangle}\langle S \cup I\rangle_{\mathbb{K}\langle \underline{x}\rangle}) \cup \{h x_i - x_i h \mid 1 \le i \le n\}\right\rangle.$$

3. We define the ordering $(\prec^E)^{\mathbf{w}}$ on $\mathrm{SMon}(\mathbb{K}\langle h, \underline{x}\rangle^E)$ for the ordering $\prec^E$ on A^E by
$$h^\alpha \underline{x}^\beta(e) (\prec^E)^{\mathbf{w}} h^{\alpha'} \underline{x}^{\beta'}(e') \text{ if and only if } \alpha + \langle \mathbf{w}, \beta\rangle < \alpha' + \langle \mathbf{w}, \beta'\rangle$$
$$\text{or } \alpha + \langle \mathbf{w}, \beta\rangle = \alpha' + \langle \mathbf{w}, \beta'\rangle \text{ and } \underline{x}^\beta(e) \prec^E \underline{x}^{\beta'}(e')$$
for $\alpha, \alpha' \in \mathbb{N}$, $\beta, \beta' \in \mathbb{N}^n$ and $e, e' \in E$.

4. We call the $\mathbb{K}$-algebra homomorphism given by
$$d_h : \mathbb{K}\langle h, \underline{x}\rangle \to \mathbb{K}\langle \underline{x}\rangle,\ h \mapsto 1, x_i \mapsto x_i$$
dehomogenization map. It induces a map $d_h : A^{\mathbf{w}} \to A$. We denote also the maps $\bigoplus_{e\in E} d_h$ by d_h.

Homogenized PBW-reduction-algebras are PBW-reduction-algebras:

Lemma 2.11. *Let $\mathbf{w} \in \mathbb{N}^n$ be a weight vector on A and set $S^{\mathbf{w}} := h_{\mathbf{w}}(S) \cup \{hx_i - x_i h \mid 1 \leq i \leq n\}$. Then $A^{\mathbf{w}} = (\mathbb{K}\langle h, \underline{x}\rangle, S^{\mathbf{w}}, I^{\mathbf{w}}, \prec^{\mathbf{w}})$ for some set $I^{\mathbf{w}} \subseteq \mathbb{K}[h, \underline{x}]$ with $(1, \mathbf{w})$-homogeneous elements. In particular, if A is a PBW-algebra, then so is $A^{\mathbf{w}}$.*

Moreover, if $\prec'$ is any ordering on A, then $(\prec')^{\mathbf{w}}$ is an ordering on $A^{\mathbf{w}}$. If $\mathbf{w}$ is strictly positive, then there exists a finite set $I'^{\mathbf{w}}$ consisting of $(1, \mathbf{w})$-homogeneous elements such that $(\mathbb{K}\langle h, \underline{x}\rangle, S^{\mathbf{w}}, I'^{\mathbf{w}}, (\prec')^{\mathbf{w}})$ is a PBW-reduction datum.

Proof. Write $S := \{x_j x_i - c_{ij} x_i x_j - d_{ij} \mid 1 \leq i < j \leq n\}$. Since $\mathbf{w}$ is a weight vector on A, we have $h_{\mathbf{w}}(x_j x_i - c_{ij} x_i x_j - d_{ij}) = x_j x_i - c_{ij} x_i x_j - h^{\alpha_{ij}} h_{\mathbf{w}}(d_{ij})$ for $1 \leq i < j \leq n$ and some $\alpha_{ij} \in \mathbb{N}$. Then $(S^{\mathbf{w}}, \prec^{\mathbf{w}})$ is a commutation system by definition of $\prec^{\mathbf{w}}$ and $A^{\mathbf{w}} = (\mathbb{K}\langle h, \underline{x}\rangle, S^{\mathbf{w}}, I^{\mathbf{w}}, \prec^{\mathbf{w}})$ is a PBW-reduction-algebra by Proposition 1.13. Using that $A^{\mathbf{w}}$ is $(1, \mathbf{w})$-graded, we replace $I^{\mathbf{w}}$ by the set of the $(1, \mathbf{w})$-homogeneous parts of its elements. The particular claim follows now from Proposition 1.11 and Definition 1.8.

As above, $(S^{\mathbf{w}}, (\prec')^{\mathbf{w}})$ is a commutation system. If $\mathbf{w}$ is strictly positive, then $(\prec')^{\mathbf{w}}$ is a well-ordering and Proposition 1.13 yields the corresponding PBW-reduction datum. □

Our approach is to homogenize A by a strictly positive weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$. This reduces Gröbner basis computations in A^E with respect to the non-well-ordering $\prec^E$ to Gröbner basis computations in $(A^{\mathbf{w}})^E$ with respect to the well-ordering $(\prec^E)^{\mathbf{w}}$. The existence of such a weight vector is guaranteed by the following lemma:

Lemma 2.12. *A weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ on A exists and is effectively computable.*

Proof. Consider the set M of standard monomials appearing with nonzero coefficient in S. According to (Greuel and Pfister, 2008, Lemma 1.2.11, Exercises 1.2.7 and 1.2.9) there is a computable $\mathbf{w} \in \mathbb{N}^n_{>0}$ such that

$$\underline{x}^\alpha \prec \underline{x}^\beta \text{ if and only if } \langle \alpha, \mathbf{w}\rangle < \langle \beta, \mathbf{w}\rangle$$

for all $\underline{x}^\alpha, \underline{x}^\beta \in M$. As $\prec$ is an ordering on A, $\mathbf{w}$ is a weight vector on A. □

If A is an elementary PBW-reduction-algebra, we compute a PBW-reduction datum for the homogenized PBW-reduction-algebra $A^{\mathbf{w}}$ with respect to the weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ as follows:

Lemma 2.13. *Consider the $\mathbb{K}$-algebra $\mathbb{K}\langle \underline{x}, \underline{y}\rangle := \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$, the elementary PBW-reduction-algebra $A = (\mathbb{K}\langle \underline{x}, \underline{y}\rangle, S, I, \prec)$ and the weight vector $\mathbf{w} \in \mathbb{N}^{n+m}_{>0}$ on A. Then $A^{\mathbf{w}}$ is an elementary PBW-reduction-algebra. In addition, if $\prec'$ is an ordering on A and $A = (\mathbb{K}\langle \underline{x}, \underline{y}\rangle, S, I', \prec'_{\mathbf{w}})$, then*

$$A^{\mathbf{w}} = (\mathbb{K}\langle h, \underline{x}, \underline{y}\rangle, S^{\mathbf{w}}, I^{\mathbf{w}}, (\prec')^{\mathbf{w}}),$$

where $I^{\mathbf{w}}$ is a Gröbner basis of $\langle h_{\mathbf{w}}(I')\rangle \subseteq \mathbb{K}[h, \underline{x}]$ with respect to the ordering induced by $(\prec')^{\mathbf{w}}$. So a PBW-reduction datum of $A^{\mathbf{w}}$ with respect to the ordering $(\prec')^{\mathbf{w}}$ is computable.

Proof. By hypothesis there is a canonical isomorphism $\psi : \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[\underline{x}]/I)\underline{y}^\beta \to A$. For the first claim we need to show that the $\mathbb{K}$-linear epimorphism

$$\psi^{\mathbf{w}} : \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[h, \underline{x}]/\langle h_{\mathbf{w}}(I)\rangle)\underline{y}^\beta \to A^{\mathbf{w}}, \ \overline{h^c \underline{x}^\alpha}\underline{y}^\beta \mapsto \overline{h^c \underline{x}^\alpha \underline{y}^\beta}$$

is injective: Consider $p = \sum_{c,\alpha,\beta} \overline{p_{c,\alpha,\beta} h^c \underline{x}^\alpha \underline{y}^\beta} \in \ker(\psi^{\mathbf{w}})$ (with $p_{c,\alpha,\beta} \in \mathbb{K}$). Because $\psi^{\mathbf{w}}$ is $(1,\mathbf{w})$-graded we may assume that $p_{c,\alpha,\beta} = 0$ for $c + \langle(\alpha,\beta),\mathbf{w}\rangle \neq k$ for some fixed $k \in \mathbb{Z}$. Define the $\mathbb{K}$-linear map

$$d_h' : \bigoplus_{\beta\in\mathbb{N}^m} (\mathbb{K}[h,\underline{x}]/\langle h_{\mathbf{w}}(I)\rangle)\underline{y}^\beta \to \bigoplus_{\beta\in\mathbb{N}^m} (\mathbb{K}[\underline{x}]/I)\underline{y}^\beta, \quad \overline{h^c\underline{x}^\alpha}\underline{y}^\beta \mapsto \overline{\underline{x}^\alpha}\underline{y}^\beta.$$

We see that $d_h \circ \psi^{\mathbf{w}} = \psi \circ d_h'$. So we obtain that $\sum_{c,\alpha} p_{c,\alpha,\beta}\underline{x}^\alpha \in I$ for all $\beta \in \mathbb{N}^m$. Since $\sum_{c,\alpha,\beta} p_{c,\alpha,\beta} h^c \underline{x}^\alpha \underline{y}^\beta$ and hence also $\sum_{c,\alpha} p_{c,\alpha,\beta} h^c \underline{x}^\alpha$ is $(1,\mathbf{w})$-homogeneous

$$\sum_{c,\alpha} p_{c,\alpha,\beta} h^c \underline{x}^\alpha = h^z h_{\mathbf{w}}\left(\sum_{c,\alpha} p_{c,\alpha,\beta}\underline{x}^\alpha\right) \in \langle h_{\mathbf{w}}(I)\rangle .$$

for some $z \in \mathbb{N}$. This implies $p = 0$ and hence that $\psi^{\mathbf{w}}$ is injective as claimed According to (Greuel and Pfister, 2008, Exercise 1.7.5) we have $h_{\mathbf{w}}(\langle I\rangle) = \langle h_{\mathbf{w}}(I')\rangle \subseteq \mathbb{K}[h,\underline{x}]$ since I' is a Gröbner basis of I with respect to $\prec_{\mathbf{w}}$. So the additional claim is immediate from Lemma 1.14. ☐

We deduce from PBW-reduction data of $A^{\mathbf{w}}$ and A a corresponding datum of the $(1,\mathbf{w})$-homogenization of a given factor algebra of A as explained below:

Lemma 2.14. *Let $\mathbf{w} \in \mathbb{N}^n_{>0}$ be a weight vector on A, $\prec'$ an ordering on A, $A^{\mathbf{w}} = (\mathbb{K}\langle h,\underline{x}\rangle, S^{\mathbf{w}}, I^{\mathbf{w}}, (\prec')^{\mathbf{w}})$ and $B = A/M$ a factor PBW-reduction-algebra. Suppose G is a Gröbner basis of M with respect to $\prec_{\mathbf{w}}$ and $G^{\mathbf{w}}$ is a Gröbner basis of the left $A^{\mathbf{w}}$-ideal generated by the residue classes of the elements in $h_{\mathbf{w}}(\tau_{A,\prec_{\mathbf{w}}}(G))$ with respect to $(\prec')^{\mathbf{w}}$. Then $\mathbf{w}$ is a weight vector on B and*

$$B^{\mathbf{w}} = (\mathbb{K}\langle h,\underline{x}\rangle, S^{\mathbf{w}}, \tau_{A^{\mathbf{w}},(\prec')^{\mathbf{w}}}(G^{\mathbf{w}}) \cup I^{\mathbf{w}}, (\prec')^{\mathbf{w}}).$$

In particular, PBW-reduction data for $\mathbf{w}$-homogenized factor algebras of PBW-algebras are computable.

Proof. Let $B = (\mathbb{K}\langle\underline{x}\rangle, S, J, \prec)$ be a PBW-reduction datum. We first show that the $\mathbb{K}$-linear morphism

$$\psi : \mathbb{K}\langle h,\underline{x}\rangle / \Big\langle h_{\mathbf{w}}({}_{\mathbb{K}\langle\underline{x}\rangle}\langle S\cup I\rangle_{\mathbb{K}\langle\underline{x}\rangle}) \cup \{hx_i - x_ih \mid 1\le i\le n\} \cup h_{\mathbf{w}}(\tau_{A,\prec_{\mathbf{w}}}(G))\Big\rangle \to B^{\mathbf{w}}, \quad \overline{p}\mapsto\overline{p}$$

is an isomorphism. Clearly, ψ is well-defined and surjective. For the injectivity let $p \in \mathbb{K}\langle h,\underline{x}\rangle$ with $\psi(\overline{p}) = 0$. Because ψ is $(1,\mathbf{w})$-graded, we may assume that p is $(1,\mathbf{w})$-homogeneous. Using the relations $hx_i - x_ih$, we may further assume that $p \in \sum_{k\ge0} h^k\,\mathbb{K}\langle\underline{x}\rangle$. By definition of $B^{\mathbf{w}}$

$$p \in {}_{\mathbb{K}\langle h,\underline{x}\rangle}\Big\langle h_{\mathbf{w}}({}_{\mathbb{K}\langle\underline{x}\rangle}\langle S\cup J\rangle_{\mathbb{K}\langle\underline{x}\rangle}) \cup \{hx_i - x_ih \mid 1\le i\le n\}\Big\rangle_{\mathbb{K}\langle h,\underline{x}\rangle}$$

and hence $d_h(p) \in {}_{\mathbb{K}\langle\underline{x}\rangle}\langle S\cup J\rangle_{\mathbb{K}\langle\underline{x}\rangle}$. Using the Gröbner basis G we find coefficients $a \in A^G$ for a standard representation

$$\overline{d_h(p)} = \sum_{g\in G} a_g g \text{ with } \operatorname{le}_{\prec_{\mathbf{w}}}(a_g) + \operatorname{le}_{\prec_{\mathbf{w}}}(g) \preceq_{\mathbf{w}} \operatorname{le}_{\prec_{\mathbf{w}}}(\overline{d_h(p)}) \preceq_{\mathbf{w}} \operatorname{le}_{\prec_{\mathbf{w}}}(d_h(p)).$$

For a suitable $r \in {}_{\mathbb{K}\langle \underline{x} \rangle}\langle S \cup I \rangle_{\mathbb{K}\langle \underline{x} \rangle}$ we obtain

$$d_h(p) = \sum_{g \in G} \tau_{A,<_{\mathbf{w}}}(a_g)\tau_{A,<_{\mathbf{w}}}(g) + r \text{ and } \operatorname{le}_{<_{\mathbf{w}}}(r) \leq_{\mathbf{w}} \operatorname{le}_{<_{\mathbf{w}}}(d_h(p)).$$

Therefore

$$p = h^{c_p} h_{\mathbf{w}}(d_h(p)) = \sum_{g \in G} h^{c'_g} h_{\mathbf{w}}(\tau_{A,<_{\mathbf{w}}}(a_g)) h_{\mathbf{w}}(\tau_{A,<_{\mathbf{w}}}(g)) + h^{c_r} h_{\mathbf{w}}(r)$$

for suitable $c' \in \mathbb{N}^{G \sqcup \{p\} \sqcup \{r\}}$ proving injectivity.

So $B^{\mathbf{w}}$ is canonically isomorphic to

$$A^{\mathbf{w}} / {}_{A^{\mathbf{w}}}\left\langle \overline{h_{\mathbf{w}}(\tau_{A,<_{\mathbf{w}}}(G))} \right\rangle_{A^{\mathbf{w}}}$$

and thus an application of Corollary 1.35 finishes the proof. □

We investigate now the relationship between $<^E$ and $(<^E)^{\mathbf{w}}$:

Remark 2.15. Let $\mathbf{w} \in \mathbb{N}^n_{>0}$ be a weight vector on A, E a finite set and $<^E$ an ordering on A^E. Then there exists for $e \in E$ a set $I^{\mathbf{w}}_e$ consisting of $(1, \mathbf{w})$-homogeneous elements such that $(<^E)^{\mathbf{w}}$ is a well-ordering on $(A^{\mathbf{w}})^E = (\mathbb{K}\langle h, \underline{x}\rangle, S^{\mathbf{w}}, I^{\mathbf{w}}_e, (<^E)^{\mathbf{w}}_e)_{e \in E}$ (see Lemma 2.11). Furthermore it holds:

1. If $\deg_{(1,\mathbf{w})}(h^\alpha \underline{x}^\beta(e)) = \deg_{(1,\mathbf{w})}(h^{\alpha'} \underline{x}^{\beta'}(e'))$, then, by definition of $(<^E)^{\mathbf{w}}$,

   $$\underline{x}^\beta(e) <^E \underline{x}^{\beta'}(e') \text{ if and only if } h^\alpha \underline{x}^\beta(e) (<^E)^{\mathbf{w}} h^{\alpha'} \underline{x}^{\beta'}(e')$$

   for $\alpha, \alpha' \in \mathbb{N}, \beta, \beta' \in \mathbb{N}^n$ and $e, e' \in E$. Thus, for any $(1, \mathbf{w})$-homogeneous $a \in \mathbb{K}[h, \underline{x}]^E$,

   $$d_h(\operatorname{lm}_{(<^E)^{\mathbf{w}}}(a)) = \operatorname{lm}_{<^E}(d_h(a)).$$

2. The map $\rho_{(A^{\mathbf{w}})^E,(<^E)^{\mathbf{w}}}$ preserves $(1, \mathbf{w})$-homogeneity since I'_e for $e \in E$ and $S^{\mathbf{w}}$ are $(1, \mathbf{w})$-homogeneous. Since the commutation relations as well as the $I^{\mathbf{w}}_e$ for $e \in E$ are $(1, \mathbf{w})$-homogeneous, Algorithm 1.31 preserves $(1, \mathbf{w})$-homogeneity.

We explain now the computation of Gröbner bases with respect to non-well-orderings.

Proposition 2.16. *Let $\mathbf{w} \in \mathbb{N}^n_{>0}$ be a weight vector on A, E a finite set, $<^E$ an ordering on A^E, and $M = {}_A\left\langle \overline{M'} \right\rangle \subseteq A^E$ for $M' \subseteq \mathbb{K}[\underline{x}]^E$ finite. If the set $G \subseteq (A^{\mathbf{w}})^E$ is a Gröbner basis of ${}_{A^{\mathbf{w}}}\left\langle \overline{h_{\mathbf{w}}(M')} \right\rangle$ with respect to $(<^E)^{\mathbf{w}}$ consisting of $(1, \mathbf{w})$-homogeneous elements, then $d_h(\tau_{(<^E)^{\mathbf{w}}}(G))$ induces a Gröbner basis of M with respect to $<^E$. An analogous statement holds for two-sided modules.*

Proof. We first show that $d_h(G) \subseteq M$: For any element $g \in G \subseteq {}_{A^{\mathbf{w}}}\left\langle \overline{h_{\mathbf{w}}(M')} \right\rangle$ there are coefficients $a \in (A^{\mathbf{w}})^{M'}$ such that $g = \sum_{m' \in M'} a_{m'} \overline{h_{\mathbf{w}}(m')}$. Hence

$$d_h(g) = \sum_{m' \in M'} d_h(a_{m'}) d_h(\overline{h_{\mathbf{w}}(m')}) = \sum_{m' \in M'} d_h(a_{m'}) \overline{m'} \in M.$$

The second step is proving that $d_h(G)$ is a Gröbner basis of M: For $t \in \mathbb{K}[\underline{x}]^E$ with $\bar{t} \in M$ there are coefficients $a \in \mathbb{K}\langle \underline{x} \rangle^{M'}$ such that $\bar{t} = \sum_{m' \in M'} \overline{a_{m'}} \overline{m'}$. This implies that there is $r \in {}_{\mathbb{K}\langle \underline{x} \rangle}\left\langle S^E \cup I^E \right\rangle_{\mathbb{K}\langle \underline{x} \rangle}$ such that $t = \sum_{m' \in M'} a_{m'} m' + r$ and hence we find $c \in \mathbb{N}^{M' \sqcup \{t\} \sqcup \{r\}}$ such that

$$h^{c_t} h_{\mathbf{w}}(t) = \sum_{m' \in M'} h^{c_{m'}} h_{\mathbf{w}}(a_{m'}) h_{\mathbf{w}}(m') + h^{c_r} h_{\mathbf{w}}(r)$$

showing that

$$\overline{h^{c_t} h_{\mathbf{w}}(t)} \in {}_{A^{\mathbf{w}}}\left\langle \overline{h_{\mathbf{w}}(M')} \right\rangle.$$

As G is a $(1, \mathbf{w})$-homogeneous Gröbner basis and $\overline{h^{c_t} h_{\mathbf{w}}(t)}$ is $(1, \mathbf{w})$-homogeneous, we obtain a $(1, \mathbf{w})[(\deg_{(1,\mathbf{w})}(g))_{g\in G}]$-homogeneous $b \in (A^{\mathbf{w}})^G$ such that

$$\overline{h^{c_t} h_{\mathbf{w}}(t)} = \sum_{g\in G} b_g g = \sum_{g\in G} \overline{\tau_{(<^E)^{\mathbf{w}}_{\mathrm{lcomp}(g)}}(b_g)} \cdot \overline{\tau_{(<^E)^{\mathbf{w}}}(g)}$$

and

$$\mathrm{le}_{(<^E)^{\mathbf{w}}_{\mathrm{lcomp}(g)}}(b_g) + \mathrm{ele}_{(<^E)^{\mathbf{w}}}(g) (\leq^E)^{\mathbf{w}} \mathrm{ele}_{(<^E)^{\mathbf{w}}}(\overline{h^{c_t} h_{\mathbf{w}}(t)}) (\leq^E)^{\mathbf{w}} \mathrm{ele}_{(<^E)^{\mathbf{w}}}(h^{c_t} h_{\mathbf{w}}(t)). \tag{18}$$

Dehomogenizing we get

$$\bar{t} = \overline{d_h(h^{c_t} h_{\mathbf{w}}(t))} = \sum_{g\in G} \overline{d_h(\tau_{(<^E)^{\mathbf{w}}_{\mathrm{lcomp}(g)}}(b_g))} \cdot \overline{d_h(\tau_{(<^E)^{\mathbf{w}}}(g))}.$$

By Equation (18) and Remark 2.15.1, we have

$$\mathrm{le}_{(<^E)^{\mathbf{w}}_{\mathrm{lcomp}(g)}}(d_h(\tau_{(<^E)^{\mathbf{w}}_{\mathrm{lcomp}(g)}}(b_g))) + \mathrm{ele}_{<^E}(d_h(\tau_{(<^E)^{\mathbf{w}}}(g))) \leq^E \mathrm{ele}_{<^E}(d_h(h^{c_t} h_{\mathbf{w}}(t))) = \mathrm{ele}_{<^E}(t)$$

concluding the proof. □

Definition 2.17. Let E a finite set and $<^E$.

1. We call a well-ordering $<^E$ on A^E *computable* if we can compute I_e for $e \in E$ such that $A^E = (\mathbb{K}\langle \underline{x} \rangle, S, I_e, <^E_e)_{e\in E}$.
2. We call the non-well-ordering $<^E$ on A^E *computable* if we can compute a weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ such that the ordering $(<^E)^{\mathbf{w}}$ on $(A^{\mathbf{w}})^E$ is computable.

Lemma 2.12, Proposition 2.16 and Remark 2.15.2 imply

Corollary 2.18. *Let E be a finite set. Gröbner bases with respect to any ordering on A^E exist and are computable for computable orderings.*

The following algorithm summarizes the computation of such Gröbner bases.

Algorithm 2.19 Given an A-submodule M of a free A-module and an ordering on that free module, this algorithm computes a Gröbner basis of M with respect to that ordering.

Input: A finite set E, an A-module $M = {}_A\left\langle \overline{M'} \right\rangle \subseteq A^E$ with $M' \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ finite and a computable ordering $<^E$ on A^E.
Output: A finite set $G \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ inducing a Gröbner basis of M with respect to $<^E$.
if $<^E$ is a well-ordering **then**
  Compute a Gröbner basis G' of M with respect to $<^E$ using Algorithm 1.31.
  return $\tau_{A^E,<^E}(G)$.
Compute a suitable weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ on A and a PBW-reduction datum for $((A^{\mathbf{w}})^E, (<^E)^{\mathbf{w}})$.
Set $M' := h_{\mathbf{w}}(M')$.

Compute a $(1, \mathbf{w})$-homogeneous Gröbner basis G' of ${}_{A^{\mathbf{w}}}\left\langle \overline{M'} \right\rangle$ over the ring $A^{\mathbf{w}}$ with respect to $(\prec^E)^{\mathbf{w}}$ using Algorithm 1.31.
Set $G := d_h(\tau_{(\prec^E)^{\mathbf{w}}}(G))$.
return G.

We can use Gröbner bases with respect to $\prec^E_{\mathbf{u}[\mathbf{s}]}$ to explicitly find generators of the filtration induced by $F^{\mathbf{u}}[\mathbf{s}]_\bullet A^E$ on submodules of A^E if that ordering is computable (see Lemma 2.9):

Algorithm 2.20 Given a weight vector $\mathbf{u}$ and an A-submodule M of a free A-module, this algorithm computes $F^{\mathbf{u}}[\mathbf{s}]_\bullet M$.

Input: A finite set E, an A-module $M = {}_A\langle M' \rangle \subseteq A^E$ with M' finite, a weight vector $\mathbf{u} \in \mathbb{Z}^n$ on A, a shift vector $\mathbf{s} \in \mathbb{Z}^E$ and a computable ordering $\prec^E_{\mathbf{u}[\mathbf{s}]}$ on A^E.
Output: A finite set $G \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ such that $F^{\mathbf{u}}[\mathbf{s}]_\bullet M = \sum_{g \in G} F^{\mathbf{u}}_{\bullet - \deg_{\mathbf{u}[s]}(g)} A \cdot \overline{g}$.
Compute a set $G \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ inducing a Gröbner basis of M with respect $\prec^E_{\mathbf{u}[\mathbf{s}]}$ by Algorithm 2.19.
return G.

Algorithm 2.21 Given a weight vector $\mathbf{u}$ and an A-submodule M of a free A-module, this algorithm computes $F^{\mathbf{u}}[\mathbf{s}]_k M$ for fixed $k \in \mathbb{Z}$.

Input: A finite set E, an A-module $M = {}_A\langle M' \rangle \subseteq A^E$ with M' finite, a weight vector $\mathbf{u} \in \mathbb{Z}^n$ on A, a shift vector $\mathbf{s} \in \mathbb{Z}^E$, a computable ordering $\prec^E_{\mathbf{u}[\mathbf{s}]}$ on A^E, and $k \in \mathbb{Z}$.
Output: A finite set $G \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ such that $F^{\mathbf{u}}[\mathbf{s}]_k M = {}_{F^{\mathbf{u}}_0 A}\langle G \rangle$.
Compute a set $G \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ inducing a Gröbner basis of M with respect $\prec^E_{\mathbf{u}[\mathbf{s}]}$ by Algorithm 2.19.
Set $G := \{a\overline{g} \mid g \in G, a \in P^{A,\mathbf{u}}_{k - \deg_{\mathbf{u}[\mathbf{s}]}(g)}\}$.
return G.

Corollary 2.18 and Lemma 2.9 imply:

Corollary 2.22. *The filtration $F^{\mathbf{u}}[\mathbf{s}]_\bullet M$ is generated by a finite set for every A-module $M \subseteq A^E$ (with E finite), every weight vector $\mathbf{s} \in \mathbb{Z}^n$ and every shift vector $\mathbf{s} \in \mathbb{Z}^E$.*

In the remainder of this subsection we aim for computing $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]} M$ as an $\mathrm{Gr}^{\mathbf{u}} A$ for an A-submodule $M \subseteq A^E$.

Proposition 2.23. *Let $\mathbf{u} \in \mathbb{Z}^n$ and $\mathbf{w} \in \mathbb{N}^n_{>0}$ be weight vectors on A, and $A^{\mathbf{w}} = (\mathbb{K}\langle h, \underline{x} \rangle, S^{\mathbf{w}}, I^{\mathbf{w}}, (\prec_{\mathbf{u}})^{\mathbf{w}})$ a PBW-reduction datum with $(1, \mathbf{w})$-homogeneous $I_{\mathbf{w}}$.*

1. *The natural $\mathbb{K}$-linear surjective map*

$$\psi : \mathbb{K}\langle \underline{x} \rangle \to \mathrm{Gr}^{\mathbf{u}} A, \; x_{i_1} \cdots x_{i_k} \mapsto \overline{x_{i_1} \cdots x_{i_k}} + F^{\mathbf{u}}_{\deg_{\mathbf{u}}(x_{i_1} \cdots x_{i_k}) - 1} A \in \mathrm{Gr}^{\mathbf{u}}_{\deg_{\mathbf{u}}(x_{i_1} \cdots x_{i_k})} A$$

identifies the $\mathbf{u}$-graded algebra associated with A with a PBW-reduction-algebra:

$$\mathrm{Gr}^{\mathbf{u}} A = \mathbb{K}\langle \underline{x} \rangle \,/\, \langle \mathrm{lt}_{\mathbf{u}}(S) \cup \mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}})) \rangle = (\mathbb{K}\langle \underline{x} \rangle, \mathrm{lt}_{\mathbf{u}}(S), \mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}})), \prec).$$

2. *If $\mathbf{u} \in \mathbb{N}^n$ and $A = (\mathbb{K}\langle \underline{x} \rangle, S, I_{\mathbf{u}}, \prec_{\mathbf{u}})$, then*

$$\mathrm{Gr}^{\mathbf{u}} A = (\mathbb{K}\langle \underline{x} \rangle, \mathrm{lt}_{\mathbf{u}}(S), \mathrm{lt}_{\mathbf{u}}(I_{\mathbf{u}}), \prec).$$

3. *Consider the finite set E, the ordering $\prec^E$ on A^E, the shift vector $\mathbf{s} \in \mathbb{Z}^E$ and the A-module $M \subseteq A^E$. Using Part (1) we can identify*

$$\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]} A^E = \mathbb{K}\langle \underline{x} \rangle^E \,/\, \left\langle \mathrm{lt}_{\mathbf{u}}(S)^E \cup \mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}}))^E \right\rangle,$$

where we put $\overline{(e)}$ in degree $\mathbf{s}_e$.

Let $G \subseteq \mathbb{K}[\underline{x}]^E$ induce a Gröbner basis of M with respect to $\prec^E_{\mathbf{u}[\mathbf{s}]}$. Under the above identification $\mathrm{lt}_{\mathbf{u}[\mathbf{s}]}(G) \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ then induces a Gröbner basis of the $\mathrm{Gr}^{\mathbf{u}} A$-submodule $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]} M$ of $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]} A^E$ with respect to $\prec^E$.

4. *Let M be a two-sided submodule of A with Gröbner basis with respect to $\prec$ induced by $G \subseteq \mathbb{K}\langle \underline{x} \rangle$. Then*

$$\mathrm{Gr}^{\mathbf{u}}(A/M) = \mathrm{Gr}^{\mathbf{u}} A / \mathrm{Gr}^{\mathbf{u}} M = (\mathbb{K}\langle \underline{x} \rangle, \mathrm{lt}_{\mathbf{u}}(S), \mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}})) \cup \rho_{\mathrm{Gr}^{\mathbf{w}} A, \prec}(\mathrm{lt}_{\mathbf{u}}(G)), \prec)$$

is a PBW-reduction-algebra.

Proof.

1. The map ψ with kernel $\left\langle \mathrm{lt}_{\mathbf{u}}({}_{\mathbb{K}\langle \underline{x} \rangle}\langle S \cup I \rangle_{\mathbb{K}\langle \underline{x} \rangle}) \right\rangle$ induces an isomorphism of $\mathbb{K}$-algebras

$$\mathbb{K}\langle \underline{x} \rangle \,/\, \left\langle \mathrm{lt}_{\mathbf{u}}({}_{\mathbb{K}\langle \underline{x} \rangle}\langle S \cup I \rangle_{\mathbb{K}\langle \underline{x} \rangle}) \right\rangle \cong \mathrm{Gr}^{\mathbf{u}} A.$$

As $d_h(I^{\mathbf{w}}) \subseteq {}_{\mathbb{K}\langle \underline{x} \rangle}\langle S \cup I \rangle_{\mathbb{K}\langle \underline{x} \rangle}$, we have $\langle \mathrm{lt}_{\mathbf{u}}(S) \cup \mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}})) \rangle \subseteq \left\langle \mathrm{lt}_{\mathbf{u}}({}_{\mathbb{K}\langle \underline{x} \rangle}\langle S \cup I \rangle_{\mathbb{K}\langle \underline{x} \rangle}) \right\rangle$. For the converse inclusion consider a $\mathbf{u}$-homogeneous $p \in {}_{\mathbb{K}\langle \underline{x} \rangle}\left\langle \mathrm{lt}_{\mathbf{u}}({}_{\mathbb{K}\langle \underline{x} \rangle}\langle S \cup I \rangle_{\mathbb{K}\langle \underline{x} \rangle}) \right\rangle_{\mathbb{K}\langle \underline{x} \rangle}$. Then there is a $p' \in \mathbb{K}\langle \underline{x} \rangle$ such that $p + p' \in {}_{\mathbb{K}\langle \underline{x} \rangle}\langle S \cup I \rangle_{\mathbb{K}\langle \underline{x} \rangle}$ and $\deg_{\mathbf{u}}(p') < \deg_{\mathbf{u}}(p)$. Using relations in S, we may assume $p, p' \in \mathbb{K}[\underline{x}]$. Now we find $l, l', l'' \in \mathbb{N}$ such that $h^{l''} h_{\mathbf{w}}(p + p') = h^l h_{\mathbf{w}}(p) + h^{l'} h_{\mathbf{w}}(p') \in {}_{\mathbb{K}\langle h, \underline{x} \rangle}\langle S^{\mathbf{w}} \cup I^{\mathbf{w}} \rangle_{\mathbb{K}\langle h, \underline{x} \rangle}$. By Lemma 1.12 we can write

$$h^{l''} h_{\mathbf{w}}(p + p') = \sum_{g \in I^{\mathbf{w}}} a_g g + \sum_{(t,s,t') \in U} t s t' \tag{19}$$

for some $(1, \mathbf{w})[(\deg_{(1,\mathbf{w})}(g))_{g \in I^{\mathbf{w}}}]$-homogeneous $a \in \mathbb{K}[h, \underline{x}]^{I^{\mathbf{w}}}$ and some finite set $U \subseteq \mathbb{K}\langle h, \underline{x} \rangle \setminus \{0\} \times S^{\mathbf{w}} \times \mathbb{K}\langle h, \underline{x} \rangle \setminus \{0\}$ such that

$$\mathrm{le}(a_g) + \mathrm{le}(g) \,(\preceq_{\mathbf{u}})^{\mathbf{w}}\, \mathrm{le}(h^{l''} h_{\mathbf{w}}(p + p'))$$

and

$$\mathrm{le}(t) + \mathrm{le}(s) + \mathrm{le}(t') \,(\preceq_{\mathbf{u}})^{\mathbf{w}}\, \mathrm{le}(h^{l''} h_{\mathbf{w}}(p + p'))$$

with equality for some $g \in I^{\mathbf{w}}$. We may assume that t and t' are $(1, \mathbf{w})$-homogeneous for all $(t, s, t') \in U$ and that all terms appearing in Equation (19) are $(1, \mathbf{w})$-homogeneous of the same degree. Dehomogenizing we obtain (see Remark 2.15(1))

$$p + p' = \sum_{g \in I^{\mathbf{w}}} d_h(a_g) d_h(g) + \sum_{(t,s,t') \in U} d_h(t) d_h(s) d_h(t')$$

with

$$\mathrm{le}(d_h(a_g)) + \mathrm{le}(d_h(g)) \preceq_{\mathbf{u}} \mathrm{le}(p + p') = \mathrm{le}(p) \tag{20}$$

and

$$\operatorname{le}(d_h(t)) + \operatorname{le}(d_h(s)) + \operatorname{le}(d_h(t')) \preceq_{\mathbf{u}} \operatorname{le}(p + p') = \operatorname{le}(p)$$

with equality for some $g \in I^{\mathbf{w}}$. By definition of $\prec_{\mathbf{u}}$ there are corresponding inequalities for the **u**-degree of the elements involved. By **u**-homogeneity of p there are $I^{\mathbf{w}\prime} \subseteq I^{\mathbf{w}}$ and $U' \subseteq U$ such that

$$p = \sum_{g \in I^{\mathbf{w}\prime}} \operatorname{lt}_{\mathbf{u}}(d_h(a_g)) \operatorname{lt}_{\mathbf{u}}(d_h(g)) + \sum_{(t,s,t') \in U'} \operatorname{lt}_{\mathbf{u}}(d_h(t)) \operatorname{lt}_{\mathbf{u}}(d_h(s)) \operatorname{lt}_{\mathbf{u}}(d_h(t')).$$

Hence $p \in \langle \mathbb{K}\langle \underline{x} \rangle \rangle \operatorname{lt}_{\mathbf{u}}(S) \cup \operatorname{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}}))$ proving the first equality.

By Remark 1.9 and the **u**-homogeneity of ${}_{\mathbb{K}\langle \underline{x} \rangle}\Big\langle \operatorname{lt}_{\mathbf{u}}({}_{\mathbb{K}\langle \underline{x} \rangle}\langle S \cup I \rangle_{\mathbb{K}\langle \underline{x} \rangle}) \Big\rangle_{\mathbb{K}\langle \underline{x} \rangle}$ it suffices to show that $\operatorname{le}_{\prec}(p) \in L_{\prec}(\operatorname{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}})))$ to obtain the second equality. To this end note that $\operatorname{le}_{\prec_{\mathbf{u}}}(r) = \operatorname{le}_{\prec_{\mathbf{u}}}(\operatorname{lt}_{\mathbf{u}}(r)) = \operatorname{le}_{\prec}(\operatorname{lt}_{\mathbf{u}}(r))$ holds for $r \in \mathbb{K}[\underline{x}]$ and thus $\operatorname{le}_{\prec}(p) = \operatorname{le}_{\prec_{\mathbf{u}}}(p)$ by **u**-homogeneity of p. Choosing $g \in I^{\mathbf{w}}$ with equality in Equation (20), we obtain

$$\operatorname{le}_{\prec}(p) = \operatorname{le}_{\prec}(\operatorname{lt}_{\mathbf{u}}(d_h(a_g))) + \operatorname{le}_{\prec}(\operatorname{lt}_{\mathbf{u}}(d_h(g))) \in L_{\prec}(\operatorname{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}}))).$$

2. Follows by similar arguments as in Part (1).

3. Consider $t \in \mathbb{K}[\underline{x}]^E$ with $0 \neq \bar{t} \in \operatorname{Gr}^{\mathbf{u[s]}} M \subseteq \mathbb{K}\langle \underline{x} \rangle^E / \Big\langle \operatorname{lt}_{\mathbf{u}}(S)^E \cup \operatorname{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}}))^E \Big\rangle$. As that module is **u[s]**-graded and the ordering $\prec^E$ is transitive, we reduce to the case that t is **u[s]**-homogeneous. Hence there exists $t' \in \mathbb{K}[\underline{x}]^E$ with $\deg_{\mathbf{u[s]}}(t') < \deg_{\mathbf{u[s]}}(t)$ such that $\overline{t + t'} \in M$. Using the Gröbner basis G, there are coefficients $a \in \mathbb{K}[\underline{x}]^G$ for a standard representation

$$\overline{t + t'} = \sum_{g \in G} \overline{a_g} \cdot \bar{g} \in M \quad \text{with} \quad \operatorname{le}_{(\prec^E_{\mathbf{u[s]}})_{\operatorname{lcomp}(g)}}(a_g) + \operatorname{le}_{\prec^E_{\mathbf{u[s]}}}(g) \preceq^E_{\mathbf{u[s]}} \operatorname{le}_{\prec^E_{\mathbf{u[s]}}}(t + t') = \operatorname{le}_{\prec^E_{\mathbf{u[s]}}}(t).$$

There is corresponding inequality of **u[s]**-degrees and we set $G' := \{g \in G \mid \deg_{\mathbf{u}}(a_g) + \deg_{\mathbf{u[s]}}(g) = \deg_{\mathbf{u[s]}}(t)\}$. Then

$$\bar{t} = \sum_{g \in G'} \overline{\operatorname{lt}_{\mathbf{u}}(a_g)} \cdot \overline{\operatorname{lt}_{\mathbf{u[s]}}(g)} \in \operatorname{Gr}^{\mathbf{u[s]}} M$$

and for $g \in G'$

$$\operatorname{le}_{\prec^E_{\operatorname{lcomp}(g)}}(\operatorname{lt}_{\mathbf{u}}(a_g)) + \operatorname{le}_{\prec^E}(\operatorname{lt}_{\mathbf{u[s]}}(g)) \preceq^E \operatorname{le}_{\prec^E}(\operatorname{lt}_{\mathbf{u[s]}}(t + t')) = \operatorname{le}_{\prec^E}(\operatorname{lt}_{\mathbf{u[s]}}(t)).$$

4. Using Corollary 1.35, the claim follows from Parts 1 and 3. □

Corollary 2.24. *If A is a PBW-algebra and $\mathbf{u} \in \mathbb{Z}^n$ a weight vector on A, then $\operatorname{Gr}^{\mathbf{u}} A$ is also a PBW-algebra.*

Proof. By Lemma 2.12 the exists a weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ on A and Lemma 2.11 implies that $A^{\mathbf{w}}$ is a PBW-algebra. Now the claim is due to Proposition 2.23.1. □

Corollary 2.25. *Consider the $\mathbb{K}$-algebra $\mathbb{K}\langle \underline{x}, \underline{y} \rangle := \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$, the elementary PBW-reduction-algebra $A = (\mathbb{K}\langle \underline{x}, \underline{y} \rangle, S, I, \prec)$ and the weight vector $\mathbf{u} \in \mathbb{Z}^{n+m}$. Then $\operatorname{Gr}^{\mathbf{u}} A$ is an elementary PBW-reduction-algebra. In addition, if I' is a Gröbner basis of ${}_{\mathbb{K}[\underline{x}]}\langle I \rangle$ with respect to the ordering $\prec_{\mathbf{u}}$ on A, then*

$$\operatorname{Gr}^{\mathbf{u}} A = (\mathbb{K}\langle \underline{x}, \underline{y} \rangle, \operatorname{lt}_{\mathbf{u}}(S), \operatorname{lt}_{\mathbf{u}}(I'), \prec).$$

Proof. Let $\mathbf{w} \in \mathbb{N}^{n+m}$ be a weight vector on A and $I^{\mathbf{w}} \subseteq \mathbb{K}[h, \underline{x}]$ a $(1, \mathbf{w})$-homogeneous Gröbner basis of $\langle h_{\mathbf{w}}(I)\rangle \subseteq \mathbb{K}[h, \underline{x}]$ with respect to the ordering induced by $(<_{\mathbf{u}})^{\mathbf{w}}$ on $\mathbb{K}[h, \underline{x}]$. Then Lemma 2.13 implies that

$$A^{\mathbf{w}} = (\mathbb{K}\langle h, \underline{x}, \underline{y}\rangle, S^{\mathbf{w}}, I^{\mathbf{w}}, (<_{\mathbf{u}})^{\mathbf{w}})$$

According to Proposition 2.23.1 it follows that $\mathrm{Gr}^{\mathbf{u}} A = (\mathbb{K}\langle \underline{x}, \underline{y}\rangle, \mathrm{lt}_{\mathbf{u}}(S), \mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}})), <)$ is an elementary PBW-reduction-algebra. By hypothesis and Proposition 2.16 both I' and $d_h(I^{\mathbf{w}})$ are a Gröbner basis of ${}_{\mathbb{K}[\underline{x}]}\langle I\rangle$ with respect to the ordering induced by $<_{\mathbf{u}}$. It follows that

$$L_<(\mathrm{lt}_{\mathbf{u}}(I')) = L_<(\mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}}))) = \mathrm{l}_<(\langle \mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}})) \cup \mathrm{lt}_{\mathbf{u}}(S)\rangle \cap \mathbb{K}[\underline{x}, \underline{y}]),$$

where the second equality follows from the above PBW-reduction datum of $\mathrm{Gr}^{\mathbf{u}} A$, and

$$\mathrm{lt}_{\mathbf{u}}(I') \subseteq \mathrm{lt}_{\mathbf{u}}({}_{\mathbb{K}[\underline{x}]}\langle I\rangle) = {}_{\mathbb{K}[\underline{x}]}\big\langle \mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}}))\big\rangle \subseteq {}_{\mathbb{K}\langle \underline{x}, \underline{y}\rangle}\big\langle \mathrm{lt}_{\mathbf{u}}(d_h(I^{\mathbf{w}})) \cup \mathrm{lt}_{\mathbf{u}}(S)\big\rangle_{\mathbb{K}\langle \underline{x}, \underline{y}\rangle}.$$

The additional claim is now due to Lemma 1.12. □

Example 2.26.

1. Consider the elementary PBW-reduction-algebra T_X introduced in Example 1.17.1 and its weight vector $\mathbf{w} = ((0)_{1\le i\le n}, (1)_{1\le i\le m})$. Then $\mathrm{Gr}^{\mathbf{w}} T_X = (\mathbb{C}\langle \underline{x}, \underline{y}\rangle, \mathrm{lt}_{\mathbf{w}}(S), I_{\mathbf{w}}, <)$ is also elementary with $\mathrm{lt}_{\mathbf{w}}(S) = \{[x_j, x_i], [y_l, y_k], [y_k, x_i] \mid 1 \le i \le j \le n, 1 \le k \le l \le m\} \setminus \{0\}$, where $I_{\mathbf{w}}$ is a Gröbner basis of I with respect to the ordering induced by $<$ on $\mathbb{C}[\underline{x}]$, since $\mathrm{lt}_{\mathbf{w}}(I) = I$ (see Corollary 2.25). In particular, $\mathrm{Gr}^{\mathbf{w}} T_X$ is a quotient algebra of the polynomial ring $\mathbb{C}[\underline{x}, \underline{y}]$. and every ordering on it is computable.

2. We have an analogous result as in Part 1 for the elementary PBW-reduction-algebra $T_X^V = (\mathbb{C}\langle \underline{x}, y_1, \ldots y_{m-1}, z\rangle, S_V, J, <')$ and the weight vector $\mathbf{w_v} = ((0)_{1\le i\le n}, (1)_{1\le i\le m})$ in the situation of Example 1.17.2: Arguing as above, we have $\mathrm{Gr}^{\mathbf{w_v}} T_X^V = (\mathbb{C}\langle \underline{x}, \underline{y}\rangle, \mathrm{lt}_{\mathbf{w}}(S_V), J_{\mathbf{w}}, <')$, where $J_{\mathbf{w}}$ is a Gröbner basis of ${}_{\mathbb{K}[\underline{x}]}\langle J\rangle$ with respect to the ordering induced by $<'$ on $\mathbb{C}[\underline{x}]$.

In algorithms we use the symbol ▷ to mark comments.

Algorithm 2.27 Given a weight vector $\mathbf{u}$ on A and an A-submodule M of a free A-module, this algorithm computes $\mathrm{Gr}^{\mathbf{u}} A$.

Input: A weight vector $\mathbf{u} \in \mathbb{Z}^n$ on A such that $<_{\mathbf{u}}$ is computable.

Output: A PBW-reduction datum $(\mathbb{K}\langle \underline{x}\rangle, \mathrm{lt}_{\mathbf{u}}(S), I_{\mathbf{u}}, <)$ of $\mathrm{Gr}^{\mathbf{u}} A$ and a finite set $G \subseteq \mathbb{K}\langle \underline{x}\rangle^E$ of $\mathbf{u}[\mathbf{s}]$-homogeneous elements whose residue classes form a set of $\mathrm{Gr}^{\mathbf{u}} A$-generators of $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]} M \subseteq \mathbb{K}\langle \underline{x}\rangle^E / \big\langle \mathrm{lt}_{\mathbf{u}}(S)^E \cup I_{\mathbf{u}}^E\big\rangle$.

if $<_{\mathbf{u}}$ is a non-well-ordering **then** ▷ Use Proposition 2.23.1
    Find a weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ such that a PBW-reduction datum $A^{\mathbf{w}} = (\mathbb{K}\langle \underline{x}\rangle, S^{\mathbf{w}}, I^{\mathbf{w}}, (<_{\mathbf{u}})^{\mathbf{w}})$ is computable.
    Replace $I^{\mathbf{w}}$ by the set of the $(1, \mathbf{w})$-homogeneous parts of its elements.
    Set $I' := d_h(I^{\mathbf{w}})$.
else
    Compute a PBW-reduction datum $(\mathbb{K}\langle \underline{x}\rangle, S, I', <_{\mathbf{u}})$ of A.
return $(\mathbb{K}\langle \underline{x}\rangle, \mathrm{lt}_{\mathbf{u}}(S), \mathrm{lt}_{\mathbf{u}[\mathbf{s}]}(I'), <)$.

Algorithm 2.28 Given a weight vector $\mathbf{u}$ on A and an A-submodule M of a free A-module, this algorithm computes $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]}M$.

Input: A weight vector $\mathbf{u} \in \mathbb{Z}^n$ on A such that $<_u$ is computable, a finite set E, an A-module $M = {}_A\left\langle \overline{M'} \right\rangle \subseteq A^E$ with $M' \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ finite and a shift vector $\mathbf{s} \in \mathbb{Z}^E$.

Output: A finite set $G \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ of $\mathbf{u}[\mathbf{s}]$-homogeneous elements whose residue classes form a set of $\mathrm{Gr}^{\mathbf{u}}A$-generators of $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]}M \subseteq \mathbb{K}\langle \underline{x} \rangle^E \,/\, \left\langle \mathrm{lt}_{\mathbf{u}}(\langle I \cup S \rangle)^E \right\rangle$.

1: Compute a finite set $G \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ inducing a Gröbner basis of M with respect to an ordering of type $(<^E, <)_{\mathbf{u}[\mathbf{s}]}$ by Algorithm 2.19.
2: Set $G := \mathrm{lt}_{\mathbf{u}[\mathbf{s}]}(G)$.
3: **return** $(\mathbb{K}\langle \underline{x} \rangle, \mathrm{lt}_{\mathbf{u}}(S), \mathrm{lt}_{\mathbf{u}[\mathbf{s}]}(I'), <)$ and G.

3. Interplay of weight filtrations and submodule structures of a free module over the PBW-reduction-algebra A

In this section, we consider two weight vectors $\mathbf{v}$ and $\mathbf{w} \in \mathbb{Z}^n$ on the PBW-reduction-algebra $A = (\mathbb{K}\langle \underline{x} \rangle, S, I, <)$, which play the role of the V- and the order filtration. We impose certain assumptions that are motivated by Hodge theory. In particular, we assume that $\mathbf{v}$ is a $\mathbf{w}$-*weight* on A, that is,

$$F_0^{\mathbf{w}}A \subseteq F_0^{\mathbf{v}}A.$$

We study the interplay of the induced weight filtrations on free A-modules with $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodule structures: Given a finite set E and $V', W' \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ finite subsets, the subjects of our investigation are the submodules

$$V := {}_{F_0^{\mathbf{v}}A}\left\langle \overline{V'} \right\rangle \subseteq A^E \text{ and } W := {}_{F_0^{\mathbf{w}}A}\left\langle \overline{W'} \right\rangle \subseteq A^E.$$

To simplify notation, we assume that $\overline{v} = \overline{v'} \in A^E$ for $v, v' \in V'$ implies $v = v'$ (and similarly for W'). For our algorithmic approach we need the following additional assumptions:

Assumption 3.1.

1. We can determine a computable ordering of type $<'_{\mathbf{v}}$ on A.
2. We can compute a PBW-reduction-datum for $F_0^{\mathbf{v}}A$. More precisely, we can determine the kernel $K_{\mathbf{v}}$ of the surjective $\mathbb{K}$-algebra map (see Notation 2.5.1)
$$\phi_{\mathbf{v}} : A_{\mathbf{v}} := \mathbb{K}\langle \{y_g \mid g \in G_A^{\mathbf{v}}\} \rangle \to F_0^{\mathbf{v}}A,\ y_g \mapsto g$$
and a PBW-reduction datum for $A_{\mathbf{v}}/K_{\mathbf{v}}$ is computable.
3. Under the assumptions of Part 2, assume in addition that the filtration $F_\bullet^{\mathbf{w}}$ induced by $F_\bullet^{\mathbf{w}}F_0^{\mathbf{v}}A$ on $A_{\mathbf{v}}/K_{\mathbf{v}}$ is given by a weight vector $\mathbf{w}_{\mathbf{v}}$ on $A_{\mathbf{v}}/K_{\mathbf{v}}$ and that we can determine a computable ordering of type $<''_{\mathbf{w}_{\mathbf{v}}}$ on $A_{\mathbf{v}}/K_{\mathbf{v}}$.
4. For any integer $d \in \mathbb{Z}$ we can determine $\mathbf{t}_d \in \mathbb{Z}^{P_d^{A,\mathbf{v}}}$ such that $F_\bullet^{\mathbf{w}}F_d^{\mathbf{v}}A = \sum_{p \in P_d^{A,\mathbf{v}}} F^{\mathbf{w}}_{\bullet-(\mathbf{t}_d)_p} F_0^{\mathbf{v}}A \cdot \overline{p}$ (see Notation 2.5.2).
5. We have $F_0^{\mathbf{v}}F_\bullet^{\mathbf{w}}A = (F_0^{\mathbf{v}}\,\mathbb{K}\langle \underline{x} \rangle \cap F_\bullet^{\mathbf{w}}\,\mathbb{K}\langle \underline{x} \rangle \cap \mathbb{K}[\underline{x}]) + \langle I \cup S \rangle) / \langle I \cup S \rangle$.

6. We can determine a computable ordering of type $<'''_{\mathbf{w}}$ for some well-ordering $<'''$ on A.

Note that Remark 2.7.2 states a sufficient condition for Assumption 3.1.4.

Remark 3.2. Given a PBW-reduction datum of A the following Gröbner basics for A-modules can be computed based on Algorithm 1.31: Gröbner bases with respect to $<$, module membership, intersections, and projections and syzygies (see Remark 1.38 and Lemma 1.37). Moreover, using Assumption 3.1 we can solve the following problems:

1. Assumption 3.1.1 enables us to compute generators of the filtration $F^{\mathbf{v}}_{\bullet}M$ for an A-submodule M of a free A-module. So in particular, we can determine $F^{\mathbf{v}}_0A$-generators of $F^{\mathbf{v}}_kM$ for $k \in \mathbb{Z}$.

2. Assumption 3.1.2 ensures that we can perform the above listed Gröbner basics also over the ring $F^{\mathbf{v}}_0A$.

3. A set of $F^{\mathbf{w}}_{\bullet}F^{\mathbf{v}}_0A$-generators of the filtration induced by $F^{\mathbf{w}}_{\bullet}A$ on $F^{\mathbf{v}}_0A$-submodules of free $F^{\mathbf{v}}_0A$-modules is computable by Assumption 3.1(3). Similarly, we will see that Assumption 3.1.5 allows us to solve the corresponding problem for $F^{\mathbf{v}}_0A$-submodules of free A-modules.

4. A computable ordering of type $<'''_{\mathbf{w}}$ on A as in Assumption 3.1.6 enables us to realize the algebra $\operatorname{Gr}^{\mathbf{w}} A$ as PBW-reduction-algebra by Algorithm 2.27.

The objective of this section is to treat the following problems:

Problem 3.3.

1. Module membership problem: Decide for $a \in A^E$ if $a \in V$ under Assumption 3.1.1 and 2.

2. Find generators of the $F^{\mathbf{w}}_0A$-module $V \cap W$ under Assumption 3.1.1-3.

3. Given that a set as in Assumption 3.1.4, show that $V \cap F^{\mathbf{w}}[\mathbf{s}]_{\bullet}A^E$ is a well-filtered $F^{\mathbf{w}}_{\bullet}F^{\mathbf{v}}_0A$-module. Compute a corresponding generating set under Assumption 3.1.1-4.

4. Under Assumption 3.1 show that $\mathbf{v}$ is a weight vector on the PBW-reduction-algebra $\operatorname{Gr}^{\mathbf{w}} A$ and represent $\operatorname{Gr}^{\mathbf{w}[\mathbf{s}]} V$ as $F^{\mathbf{v}}_0 \operatorname{Gr}^{\mathbf{w}} A$-module.

Remark 3.4. In case $\mathbf{v} = (0)_{1\le i\le n}$, $F^{\mathbf{v}}_0A = A$ and Problem 3.3.2 deals with the intersection of an A-submodule M of A^E with a finitely generated $F^{\mathbf{u}}_0A$-submodule of A^E.

Example 3.5. With regard to our applications to Hodge theory, we are particularly interested in the situation of Example 1.17 in the case

$$\mathbf{v} = ((-\delta_{n,i})_{1\le i\le n}, (\delta_{m,i})_{1\le i\le m}) \in \mathbb{Z}^{n+m} \text{ and } \mathbf{w} = ((0)_{1\le i\le n}, (1)_{1\le i\le m}) \in \mathbb{Z}^{n+m}$$

under the condition that x_n is a local coordinate (see Example 1.17.2). In this case, $F^{\mathbf{v}}_{\bullet}T_X$ is the so-called V-filtration on $D_X(X)$ with respect to the divisor $\{x_n = 0\}$ and $F^{\mathbf{w}}_{\bullet}A$ is the filtration with respect to the order of differential operators on $D_X(X)$.

Note that we can indeed determine a PBW-reduction datum for T_X by Example 1.17.1. Moreover Assumption 3.1 is satisfied: Part 1 follows by Lemma 2.12 and Lemma 2.13. For Part 2 recall that $F^{\mathbf{v}}_0T_X$ is isomorphic to the elementary PBW-reduction-algebra T^V_X by Example 1.17.2.

By Lemma 1.14 a corresponding PBW-reduction datum can be computed. By Example 2.8 we know that $\mathbf{w}$ induces the weight vector $\mathbf{w}_{\mathbf{v}} = ((0)_{1\le i\le n}, (1)_{1\le i\le m})$ on T_X^V. Again by Lemma 1.14 this show that Part 3 is satisfied. With $P_d^{T_X,\mathbf{v}}$ as in example, that Remark 2.7.1 and 2 yields

$$F_\bullet^{\mathbf{w}} F_d^{\mathbf{v}} T_X = \begin{cases} F_\bullet^{\mathbf{w}} F_0^{\mathbf{v}} T_X \cdot \overline{x_n^d} & \text{if } d \le 0, \\ \sum_{0\le l\le d} F_{\bullet-l}^{\mathbf{w}} F_0^{\mathbf{v}} T_X \cdot \overline{y_m^l} & \text{otherwise,} \end{cases}$$

and Part 4 is satisfied. Remark 2.7.1 shows that also Part 5 holds in this situation. Finally Part 6 is an immediate consequence of Lemma 1.14.

Part of the difficulty of the above problems is due to module structures over different subrings in the chain of non-finite ring extensions $F_0^{\mathbf{w}} A \subseteq F_0^{\mathbf{v}} A \subseteq A$.

3.1. A one-to-one correspondence for $F_0^{\mathbf{w}}A$-submodules of bounded $\mathbf{v}$-degree of a free A-module

Thus we first reduce to a problem involving only the PBW-reduction-algebra $F_0^{\mathbf{v}}A$ and its subalgebra $F_0^{\mathbf{w}}A$. To this end we consider $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodules of A^E of $\mathbf{v}$-degree bounded by d and lift them to a presentation of the $F_0^{\mathbf{v}}A$-module $F_d^{\mathbf{v}}A^E$.

Remark 3.6. The inclusion $F_0^{\mathbf{w}}A \subseteq F_0^{\mathbf{v}}A$ implies that for any finite set $N' \subseteq A^E$

$$\deg_{\mathbf{v}}({}_{F_0^{\mathbf{v}}A}\langle N'\rangle) = \deg_{\mathbf{v}}({}_{F_0^{\mathbf{w}}A}\langle N'\rangle) = \deg_{\mathbf{v}}(N') < \infty.$$

To construct the above presentation take $F_0^{\mathbf{v}}A$-generators $P_d^{A,\mathbf{v}}$ of $F_d^{\mathbf{v}}A$ (see Notation 2.5(2)) and consider the $F_0^{\mathbf{v}}A$-linear surjective map

$$\omega_{\mathbf{v},d} : F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}} \to F_d^{\mathbf{v}}A,\ q \mapsto \sum_{p\in P_d^{A,\mathbf{v}}} q_p \overline{p}. \tag{21}$$

Consider $F_0^{\mathbf{v}}A$-generators $K_{\omega_{\mathbf{v},d}}$ of $\ker(\omega_{\mathbf{v},d}) = \operatorname{syz}_A(\overline{P_d^{A,\mathbf{v}}}) \cap F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}}$ as can be computed by Algorithm 2.21 under Assumption 3.1.1. For every $a \in F_d^{\mathbf{v}}A$ fix a representation

$$a = \sum_{p\in P_d^{A,\mathbf{v}}} q_p^a \overline{p} \text{ with } q^a \in F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}}, \tag{22}$$

computable by Remark 2.6, to define a right inverse map of $\omega_{\mathbf{v},d}$

$$\upsilon_{\mathbf{v},d} : F_d^{\mathbf{v}}A \to F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}},\ a \mapsto q^a. \tag{23}$$

The following one-to-one correspondence is now an immediate consequence of the homomorphism theorem:

Lemma 3.7. *Let $d \in \mathbb{Z}$. There is an inclusion-, intersection- and sum-preserving one-to-one correspondence*

$$\begin{aligned} \{F_0^{\mathbf{w}}A\text{-modules } K \subseteq (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E \mid \ker(\omega_{\mathbf{v},d}^E) \subseteq K\} &\leftrightarrow \{F_0^{\mathbf{w}}A\text{-modules } J \subseteq F_d^{\mathbf{v}}A^E\} \\ \Omega_{\mathbf{v},d}^E : \quad K &\mapsto \omega_{\mathbf{v},d}^E(K) \\ \upsilon_{\mathbf{v},d}^E(J) + \ker(\omega_{\mathbf{v},d}^E) &\leftarrow J \quad : \Upsilon_{\mathbf{v},d}^E. \end{aligned}$$

It identifies $F_0^{\mathbf{v}}A$-modules on both sides. Moreover, if $K' \subseteq F_d^{\mathbf{v}}A^E$ and $\mathbf{u} \in \{\mathbf{v}, \mathbf{w}\}$, then

$$\Upsilon_{\mathbf{v},d}^E({}_{F_0^{\mathbf{u}}A}\langle K'\rangle) = {}_{F_0^{\mathbf{u}}A}\left\langle \upsilon_{\mathbf{v},d}^E(K')\right\rangle + \ker(\omega_{\mathbf{v},d}^E).$$

The following algorithms compute images of $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodules under the above one-to-one correspondence.

Algorithm 3.8 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{w}}A$-submodule $M \subseteq A^E$, this algorithm computes $\upsilon^E_{\mathbf{v},d}(M)$ for some $d \geq \deg_{\mathbf{v}}(M)$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight, a finite set E, a computable ordering of type $<'_{\mathbf{v}}$ on A, a finite set $M \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ and an optional natural number d'.

Output: Two finite subsets $M', K \subseteq (F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}})^E$, such that $Y^E_{\mathbf{v},d}({}_{F_0^{\mathbf{u}}A}\langle \overline{M} \rangle) = {}_{F_0^{\mathbf{u}}A}\langle M' \rangle + {}_{F_0^{\mathbf{v}}A}\langle K \rangle$ for $\mathbf{u} \in \{\mathbf{v}, \mathbf{w}\}$ and $\ker(\omega^E_{\mathbf{v},d}) = {}_{F_0^{\mathbf{v}}A}\langle K \rangle$, where $d := \max\{\deg_{\mathbf{v}}(M)[, d']\}$.

1: Set $d := \max\{\deg_{\mathbf{v}}(M)[, d']\}$ and determine $P_d^{A,\mathbf{v}}$.
2: $M' := \emptyset$.
3: **for** $m \in M$ **do**
4: Find $q^m \in (F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}})^E$ such that $\overline{m} = \sum_{e \in E} \sum_{p \in P_d^{A,\mathbf{v}}} q^m_{e_p} \overline{p}(e)$ as explained in Remark 2.6.
5: $M' := M' \cup \{q^m\}$.
6: Compute $F_0^{\mathbf{v}}A$-generators K of $\operatorname{syz}_A(\overline{P_d^{A,\mathbf{v}}}) \cap F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}}$ by Algorithm 2.21 using the ordering $(<'_{\mathbf{v}}, <^{P_d^{A,\mathbf{v}}})$ for some order $<^{P_d^{A,\mathbf{v}}}$ on $P_d^{A,\mathbf{v}}$.
7: **return** M', K^E.

In the above algorithm, we mean by $\max\{\deg_{\mathbf{v}}(M)[, d']\}$ the value $\max\{\deg_{\mathbf{v}}(M), d'\}$ if d' is defined and $\deg_{\mathbf{v}}(M)$ otherwise.

Algorithm 3.9 Given a weight vector $\mathbf{v}$ on A and a subset $M \subseteq (F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}})^E$, this algorithm computes $\omega^E_{\mathbf{v},d}(M)$.

Input: A weight vector $\mathbf{v} \in \mathbb{Z}^n$ on A, an integer $d \in \mathbb{Z}$, a finite set E and a finite subset $M \subseteq (F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}})^E$.

Output: A set $M' \subseteq A^E$ such that $\omega^E_{\mathbf{v},d}(M) = M'$.

1: Set $M' := \emptyset$.
2: **for** $m \in M$ **do**
3: $M' := M' \cup \{\sum_{e \in E} \sum_{p \in P_d^{A,\mathbf{v}}} m_{e_p} \overline{p}(e)\}$.
4: **return** M'.

3.2. Module membership problem for $F_0^{\mathbf{v}}A$-submodules of free A-modules

In this subsection, we require that Assumption 3.1.1 and 2 is satisfied. Recall that $V = {}_{F_0^{\mathbf{v}}A}\langle \overline{V'} \rangle \subseteq A^E$ with $V' \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ finite and consider $a \in \mathbb{K}\langle \underline{x} \rangle^E$. We explain how to check whether $\overline{a} \in V$, which is equivalent to ${}_{F_0^{\mathbf{v}}A}\langle \overline{a} \rangle \subseteq V$. Since the $\mathbf{v}$-degree of the above ideals is bounded by $d := \max\{\deg_{\mathbf{v}}(V'), \deg_{\mathbf{v}}(a)\}$ and the one-to-one correspondence in Lemma 3.7 is inclusion-preserving, our problem reduces to deciding whether

$$
{}_{F_0^{\mathbf{v}}A}\left\langle \upsilon^E_{\mathbf{v},d}(\overline{a}) \right\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}} \right\rangle \subseteq {}_{F_0^{\mathbf{v}}A}\left\langle \upsilon^E_{\mathbf{v},d}(\overline{V'}) \right\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}} \right\rangle,
$$

which is in turn equivalent to

$$
\upsilon^E_{\mathbf{v},d}(\overline{a}) \in {}_{F_0^{\mathbf{v}}A}\left\langle \upsilon^E_{\mathbf{v},d}(\overline{V'}) \cup K^E_{\omega_{\mathbf{v},d}} \right\rangle.
$$

The above module membership problem can be solved over the PBW-reduction-algebra $F_0^{\mathbf{v}}A$ by a normal form computation. The following algorithm checks more generally whether ${}_{F_0^{\mathbf{v}}A}\langle P\rangle \subseteq V$ for $P \subseteq A^E$ finite.

Algorithm 3.10 Given a weight vector $\mathbf{v}$ on A and two $F_0^{\mathbf{v}}A$-submodules V, P of a free A-module, this algorithm checks if $P \subseteq V$.

Input: A weight vector $\mathbf{v} \in \mathbb{Z}^n$ on A, such that Assumption 3.1.1 and 2 is satisfied, a finite set E and submodules $V := {}_{F_0^{\mathbf{v}}A}\left\langle \overline{V'} \right\rangle$, $P := {}_{F_0^{\mathbf{v}}A}\left\langle \overline{P'} \right\rangle \subseteq A^E$ with $V', P' \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ finite.

Output: `true` if $P \subseteq V$ and `false` otherwise.

Set $d := \max\{\deg_{\mathbf{v}}(V'), \deg_{\mathbf{v}}(P')\}$.
Compute $P'' := \upsilon_{\mathbf{v},d}^E(\overline{P'})$, $V'' := \upsilon_{\mathbf{v},d}^E(\overline{V'})$ and $K := K_{\omega_{\mathbf{v},d}}^E$ using Algorithm 3.8.
Set $J := {}_{F_0^{\mathbf{v}}A}\langle V'' \cup K\rangle$.
for $p'' \in P''$ **do**
    if $p'' \notin J$ **then** ▷ Use Algorithm 1.26 over the PBW-reduction-algebra $F_0^{\mathbf{v}}A$.
        return `false`.
return `true`.

Remark 3.11. With a little extra bookkeeping the above algorithm can be extended to represent $\overline{p'}$ for $p' \in P'$ as an $F_0^{\mathbf{v}}A$-linear combination of the elements of $\overline{V'}$ if $\overline{p'} \in V$.

3.3. Intersection of $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodules of a free A-module

In this subsection, we require that Assumption 3.1.1-3 is satisfied. Based on the one-to-one correspondence of Subsection 3.1 we describe a method to compute generators the $F_0^{\mathbf{w}}A$-submodule

$$V \cap W \subseteq A^E,$$

where $V = {}_{F_0^{\mathbf{v}}A}\left\langle \overline{V'} \right\rangle$ and $W = {}_{F_0^{\mathbf{w}}A}\left\langle \overline{W'} \right\rangle$. Setting $d := \max\{\deg_{\mathbf{v}}(V'), \deg_{\mathbf{v}}(W')\} \in \mathbb{Z}$, we get by the one-to-one correspondence in Lemma 3.7 and Remark 3.6

$$V \cap W = \omega_{\mathbf{v},d}^E(J_W \cap J_V),$$

where

$$J_W = {}_{F_0^{\mathbf{w}_{\mathbf{v}}}F_0^{\mathbf{v}}A}\left\langle \upsilon_{\mathbf{v},d}^E(\overline{W'}) \right\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K_{\omega_{\mathbf{v},d}}^E \right\rangle \tag{24}$$

and

$$J_V = {}_{F_0^{\mathbf{v}}A}\left\langle \upsilon_{\mathbf{v},d}^E(\overline{V'}) \right\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K_{\omega_{\mathbf{v},d}}^E \right\rangle. \tag{25}$$

To ease notation we identify $F_0^{\mathbf{v}}A^{W'} = F_0^{\mathbf{v}}A^{\upsilon_{\mathbf{v},d}^E(\overline{W'})}$ and $F_0^{\mathbf{v}}A^{V'} = F_0^{\mathbf{v}}A^{\upsilon_{\mathbf{v},d}^E(\overline{V'})}$ and set $K := K_{\omega_{\mathbf{v},d}}^E$. Now consider the syzygy module

$$R := \operatorname{syz}_{F_0^{\mathbf{v}}A}\left(\upsilon_{\mathbf{v},d}^E(\overline{W'}), \upsilon_{\mathbf{v},d}^E(\overline{V'}), K\right) \subseteq F_0^{\mathbf{v}}A^{W'} \oplus F_0^{\mathbf{v}}A^{V'} \oplus F_0^{\mathbf{v}}A^{K} \xrightarrow{\pi_{W'}} F_0^{\mathbf{v}}A^{W'}$$

and set

$$R' := \pi_{W'}(R) \cap F_0^{\mathbf{w}_{\mathbf{v}}}F_0^{\mathbf{v}}A^{W'}.$$

A set of $F_0^{\mathbf{v}}A$-generators of R can be obtained from a Gröbner basis calculation over the PBW-reduction-algebra $F_0^{\mathbf{v}}A$ (see Lemma 1.37). By Algorithm 2.21 we can determine a finite set G such that $R' = {}_{F_0^{\mathbf{w}_{\mathbf{v}}}F_0^{\mathbf{v}}A}\langle G\rangle$. The intersection $V \cap W$ is obtained from G as follows.

Lemma 3.12. *We have*

$$J_W \cap J_V = {}_{F_0^{\mathbf{w}_\mathbf{v}} F_0^{\mathbf{v}} A}\left\langle \{ \sum_{w' \in W'} g_{w'} \upsilon_{\mathbf{v},d}^E(\overline{w'}) \mid g \in G \} \right\rangle + {}_{F_0^{\mathbf{v}} A}\langle K \rangle \tag{26}$$

and hence $V \cap W = {}_{F_0^{\mathbf{w}} A}\left\langle \{ \sum_{w' \in W'} g_{w'} \overline{w'} \mid g \in G \} \right\rangle$.

Proof. For the non-trivial inclusion of Equation (26) pick $q \in J_W \cap J_V$. Then there exist $a \in F_0^{\mathbf{w}_\mathbf{v}} F_0^{\mathbf{v}} A^{W'}$, $b \in F_0^{\mathbf{v}} A^{V'}$ and $c, c' \in F_0^{\mathbf{v}} A^K$ such that

$$q = \sum_{w' \in W'} a_{w'} \upsilon_{\mathbf{v},d}^E(\overline{w'}) + \sum_{k \in K} c_k k = \sum_{v' \in V'} b_{v'} \upsilon_{\mathbf{v},d}^E(\overline{v'}) + \sum_{k \in K} c'_k k.$$

This implies that $(a, -b, c - c') \in R$. By the choice of G, there is $f \in F_0^{\mathbf{w}_\mathbf{v}} F_0^{\mathbf{v}} A^G$ such that $a = \sum_{g \in G} f_g g$ and hence $\sum_{w' \in W'} a_{w'} \upsilon_{\mathbf{v},d}^E(\overline{w'}) = \sum_{g \in G} f_g \sum_{w' \in W'} g_{w'} \upsilon_{\mathbf{v},d}^E(\overline{w'})$, which is in the right hand side of Equation (26). The second equality follows immediately. □

Algorithm 3.13 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A, an $F_0^{\mathbf{v}} A$-submodule V and an $F_0^{\mathbf{w}} A$-submodule W of a free A-module, this algorithm computes the intersection $V \cap W$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 3.1.1-3 is satisfied, a finite set E, submodules $V := {}_{F_0^{\mathbf{v}} A}\left\langle \overline{V'} \right\rangle$, $W := {}_{F_0^{\mathbf{w}} A}\left\langle \overline{W'} \right\rangle \subseteq A^E$ with $V', W' \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ finite.

Output: A finite set $G \subseteq A^E$ such that ${}_{F_0^{\mathbf{w}} A}\langle G \rangle = V \cap W$.

1: Set $d := \max\{\deg_{\mathbf{v}}(V'), \deg_{\mathbf{v}}(W')\}$.
2: Compute $V'' := \upsilon_{\mathbf{v},d}^E(\overline{V'})$, $W'' := \upsilon_{\mathbf{v},d}^E(\overline{W'})$ and $K := K_{\omega_{\mathbf{v},d}}^E$ by Algorithm 3.8.
3: Compute $R := \operatorname{syz}_{F_0^{\mathbf{v}} A}(W'', V'', K) \subseteq F_0^{\mathbf{v}} A^{W'} \oplus F_0^{\mathbf{v}} A^{V'} \oplus F_0^{\mathbf{v}} A^K$ using Algorithm 1.31 with the setup of Lemma 1.37 over the PBW-reduction-algebra $F_0^{\mathbf{v}} A$.
4: Determine G' such that ${}_{F_0^{\mathbf{w}_\mathbf{v}} F_0^{\mathbf{v}} A}\langle G' \rangle = \pi_{W'}(R) \cap F_0^{\mathbf{w}_\mathbf{v}} F_0^{\mathbf{v}} A^{W'}$ using Algorithm 2.21 over $F_0^{\mathbf{v}} A$.
5: Set $G := \{ \sum_{w' \in W'} g'_{w'} \overline{w'} \mid g' \in G' \}$.
6: **return** G.

Remark 3.14. By setting $\mathbf{w} := \mathbf{v}$, Algorithm 3.13 enables us to determine the intersection of finitely generated $F_0^{\mathbf{v}} A$-modules. In this case, we do not need to apply Algorithm 2.21.

3.4. Induced $\mathbf{w}$*-weight filtration on* $F_0^{\mathbf{v}} A$*-submodules of free* A*-modules*

In this subsection, we require that Assumption 3.1.1-4 is satisfied. We explain how to compute $F_\bullet^{\mathbf{w}} F_0^{\mathbf{v}} A$-generators of the module

$$F^{\mathbf{w}}[\mathbf{s}]_\bullet V = V \cap F^{\mathbf{w}}[\mathbf{s}]_\bullet A^E,$$

where $V = {}_{F_0^{\mathbf{v}} A}\left\langle \overline{V'} \right\rangle$ with $V' \subseteq \mathbb{K}\langle \underline{x} \rangle^E$ finite. Bounding the $\mathbf{v}$-degree by $d := \deg_{\mathbf{v}}(V')$, we proceed as in Subsection 3.3. By Assumption 3.1.4 there are $P_d^{A,\mathbf{v}}$ and $\mathbf{t}_d \in \mathbb{Z}^{P_d^{A,\mathbf{v}}}$ such that $F_\bullet^{\mathbf{w}} F_d^{\mathbf{v}} A = \sum_{p \in P_d^{A,\mathbf{v}}} F_{\bullet - (\mathbf{t}_d)_p}^{\mathbf{w}} F_0^{\mathbf{v}} A \cdot \overline{p}$. We get by Lemma 3.7 and Remark 3.6

$$F^{\mathbf{w}}[\mathbf{s}]_\bullet V = V \cap F^{\mathbf{w}}[\mathbf{s}]_\bullet F_d^{\mathbf{v}} A^E = \omega_{\mathbf{v},d}^E(J_V \cap J_{F^{\mathbf{w}}[\mathbf{s}]_\bullet}),$$

where

$$J_V = {}_{F_0^{\mathbf{v}}A}\left\langle \upsilon_{\mathbf{v},d}^E(\overline{V'})\right\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K_{\omega_{\mathbf{v},d}}^E\right\rangle \quad \text{and} \quad J_{F^{\mathbf{w}}[\mathbf{s}]_\bullet} = F^{\mathbf{w}_{\mathbf{v}}}[\mathbf{t}]_\bullet (F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}})^E + {}_{F_0^{\mathbf{v}}A}\left\langle K_{\omega_{\mathbf{v},d}}^E\right\rangle$$

with $\mathbf{t}_{e_p} = \mathbf{s}_e + (\mathbf{t}_d)_p$ for $e \in E$, $p \in P_d^{A,\mathbf{v}}$. It follows that

$$J_V \cap J_{F^{\mathbf{w}}[\mathbf{s}]_\bullet} = \left(J_V \cap F^{\mathbf{w}_{\mathbf{v}}}[\mathbf{t}]_\bullet (F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}})^E\right) + {}_{F_0^{\mathbf{v}}A}\left\langle K_{\omega_{\mathbf{v},d}}^E\right\rangle.$$

Applying Algorithm 2.20 over $F_0^{\mathbf{v}}A$, we determine a finite set $G \subseteq (\mathbb{K}\langle \underline{x}\rangle^{P_d^{A,\mathbf{v}}})^E$ such that

$$J_V \cap F^{\mathbf{w}_{\mathbf{v}}}[\mathbf{t}]_\bullet (F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}})^E = \sum_{g\in G} F^{\mathbf{w}_{\mathbf{v}}}_{\bullet - \deg_{\mathbf{w}_{\mathbf{v}}[\mathbf{t}]}(g)} F_0^{\mathbf{v}}A \cdot \overline{g}.$$

Since $\deg_{\mathbf{w}[\mathbf{s}]}(\omega_{\mathbf{v},d}^E(\overline{g})) \le \deg_{\mathbf{w}_{\mathbf{v}}[\mathbf{t}]}(\overline{g}) \le \deg_{\mathbf{w}_{\mathbf{v}}[\mathbf{t}]}(g)$ by Assumption 3.1.4, this implies that

$$F^{\mathbf{w}}[\mathbf{s}]_\bullet V = \sum_{g\in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}_{\mathbf{v}}[\mathbf{t}]}(g)} F_0^{\mathbf{v}}A \cdot \omega_{\mathbf{v},d}^E(\overline{g}) = \sum_{g\in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(\omega_{\mathbf{v},d}^E(\overline{g}))} F_0^{\mathbf{v}}A \cdot \omega_{\mathbf{v},d}^E(\overline{g}).$$

Algorithm 3.15 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{v}}A$-submodule V of a free A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{w}}[\mathbf{s}]_\bullet V$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 3.1.1-4 is satisfied, a finite set E, a submodule $V := {}_{F_0^{\mathbf{v}}A}\left\langle \overline{V'}\right\rangle \subseteq A^E$ with $V' \subseteq \mathbb{K}\langle \underline{x}\rangle^E$ finite and a shift vector $\mathbf{s} \in \mathbb{Z}^E$.

Output: A finite set $G \subseteq A^E$ and $\mathbf{t} \in \mathbb{Z}^G$ which satisfy $F^{\mathbf{w}}[\mathbf{s}]_\bullet V = \sum_{g\in G} F^{\mathbf{w}}_{\bullet - \mathbf{t}_g} F_0^{\mathbf{v}}A \cdot g = \sum_{g\in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}}A \cdot g$.

1: Set $d := \deg_{\mathbf{v}}(V')$.
2: Choose $P_d^{A,\mathbf{v}}$ and $\mathbf{t}_d \in \mathbb{Z}^{P_d^{A,\mathbf{v}}}$ such that $F_\bullet^{\mathbf{w}} F_d^{\mathbf{v}}A = \sum_{p\in P_d^{A,\mathbf{v}}} F^{\mathbf{w}}_{\bullet - (\mathbf{t}_d)_p} F_0^{\mathbf{v}}A \cdot \overline{p}$.
3: Compute $V'' := \upsilon_{\mathbf{v},d}^E(\overline{V'})$ and $K := K_{\omega_{\mathbf{v},d}}^E$ using Algorithm 3.8.
4: Define the shift vector $\mathbf{t} \in (\mathbb{Z}^{P_d^{A,\mathbf{v}}})^E$ by $\mathbf{t}_{e_p} = \mathbf{s}_e + (\mathbf{t}_d)_p$ for $e \in E$ and $p \in P_d^{A,\mathbf{v}}$.
5: Find $G' \subseteq (\mathbb{K}\langle \underline{x}\rangle^{P_d^{A,\mathbf{v}}})^E$ such that $\sum_{g'\in G'} F^{\mathbf{w}_{\mathbf{v}}}_{\bullet - \deg_{\mathbf{w}_{\mathbf{v}}[\mathbf{t}]}(g')} F_0^{\mathbf{v}}A \cdot \overline{g'} = {}_{F_0^{\mathbf{v}}A}\langle V'' \cup K\rangle \cap F^{\mathbf{w}_{\mathbf{v}}}[\mathbf{t}]_\bullet (F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}})^E$ using Algorithm 2.20 over $F_0^{\mathbf{v}}A$.
6: Define $\mathbf{t}' \in \mathbb{Z}^{G'}$ by $\mathbf{t}'_g := \deg_{\mathbf{w}_{\mathbf{v}}[\mathbf{t}]}(g')$ for $g \in G'$.
7: Compute $G := \omega_{\mathbf{v},d}^E(\overline{G'})$ by applying Algorithm 3.9 and define $\mathbf{t}'' \in \mathbb{Z}^G$ by $\mathbf{t}''_g := \min\{\mathbf{t}'_{g'} \mid g' \in G' \text{ with } \omega_{\mathbf{v},d}^E(\overline{g'}) = g\}$.
8: **return** $G, \mathbf{t}''$.

Remark 3.16. The above algorithm implicitly computes a representative $g' \in \mathbb{K}\langle \underline{x}\rangle^E$ of $g \in G$ with $\deg_{\mathbf{w}[\mathbf{s}]}(g') \le \mathbf{t}_g$

3.5. Associated $F_\bullet^{\mathbf{w}}$-graded modules to $F_0^{\mathbf{v}}A$-submodules of free A-modules

In this subsection we require Assumption 3.1. We explain how to express $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]} V$ for $V = {}_{F_0^{\mathbf{v}}A}\langle V'\rangle$ as a finitely generated $F_0^{\mathbf{v}}\,\mathrm{Gr}^{\mathbf{w}}A$-module.

Proposition 3.17. *Let $\mathbf{s} \in \mathbb{Z}^E$ be a shift vector and $\mathrm{Gr}^{\mathbf{w}}A = (\mathbb{K}\langle \underline{x}\rangle, \mathrm{lt}_{\mathbf{w}}(S), I_{\mathbf{w}}, \prec)$ under the identification made in Proposition 2.23.1.*

1. *The vector* $\mathbf{v}$ *is a weight vector on* $(\mathbb{K}\langle\underline{x}\rangle, \mathrm{lt}_{\mathbf{w}}(S), I_{\mathbf{w}}, <)$. *With* $F^{\mathbf{v}}_{\bullet}\,\mathrm{Gr}^{\mathbf{w}}A$ *induced by* $F^{\mathbf{v}}_{\bullet}A$

$$F^{\mathbf{v}}_0\,\mathrm{Gr}^{\mathbf{w}}A = F^{\mathbf{v}}_0(\mathbb{K}\langle\underline{x}\rangle \,/\, \langle \mathrm{lt}_{\mathbf{w}}(S) \cup I_{\mathbf{w}}\rangle).$$

2. *We may consider* $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}\,V$ *as an* $F^{\mathbf{v}}_0\,\mathrm{Gr}^{\mathbf{w}}\,A$*-submodule of* $(\mathrm{Gr}^{\mathbf{w}}\,A)^E = \mathbb{K}\langle\underline{x}\rangle^E \,/\, \left\langle \mathrm{lt}_{\mathbf{w}}(S)^E \cup I^E_{\mathbf{w}}\right\rangle$, *where we put* $\overline{(e)}$ *in degree* $\mathbf{s}_e$. *If* $G \subseteq \mathbb{K}\langle\underline{x}\rangle^E$ *is finite with* $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V = \sum_{g\in G} F^{\mathbf{w}}_{\bullet-\deg_{\mathbf{w}[\mathbf{s}]}(g)}F^{\mathbf{v}}_0 A\cdot \overline{g}$, *then* $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}\,V$ *is* $F^{\mathbf{v}}_0\,\mathrm{Gr}^{\mathbf{w}}A$*-generated by* $\overline{\mathrm{lt}_{\mathbf{w}[\mathbf{s}]}(G)}$ *under the above identification.*

Proof.

1. Since $\mathbf{v}$ and $\mathbf{w}$ are weight vectors on A, $\mathbf{v}$ is a one on $\mathrm{Gr}^{\mathbf{w}}A$. The map ψ in Proposition 2.23.1 induces the claimed equality by Assumption 3.1.5.

2. The identification follows from $\mathrm{Gr}^{\mathbf{w}}_k\,F^{\mathbf{v}}_0 A = F^{\mathbf{v}}_0\,\mathrm{Gr}^{\mathbf{v}}_k\,A$ for $k \in \mathbb{Z}$ combined with Part 1. To prove the statement on generation let $\overline{v} \in F^{\mathbf{w}}[\mathbf{s}]_k V$ with $v \in F^{\mathbf{w}}[\mathbf{s}]_k\,\mathbb{K}\langle\underline{x}\rangle^E$ represent a non-zero element v' in $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}_k\,V$. By hypothesis there exits $f \in F^{\mathbf{v}}_0\,\mathbb{K}\langle\underline{x}\rangle^G$ with $f_g \in F^{\mathbf{w}}_{k-\deg_{\mathbf{w}[\mathbf{s}]}(g)}\,\mathbb{K}\langle\underline{x}\rangle$ for all $g \in G$ such that $\overline{v} = \sum_{g\in G}\overline{f_g}\overline{g}$ and hence

$$v = \sum_{g\in G} f_g g + p$$

for some $p \in F^{\mathbf{w}}[\mathbf{s}]_k\,\langle I \cup S\rangle$. Taking $\mathbf{w}[\mathbf{s}]$-leading terms v' identifies with

$$\overline{\mathrm{lt}_{\mathbf{w}[\mathbf{s}]}(v)} = \sum_{g\in G'} \overline{\mathrm{lt}_{\mathbf{w}}(f_g)}\cdot\overline{\mathrm{lt}_{\mathbf{w}[\mathbf{s}]}(g)} \in \mathbb{K}\langle\underline{x}\rangle^E \,/\, \left\langle I^E_{\mathbf{w}} \cup \mathrm{lt}_{\mathbf{w}}(S)^E\right\rangle$$

for a suitable subset $G' \subseteq G$. □

Note that Assumption 3.1.1-4 enables us to find G as in the above proposition yielding the following algorithm:

Algorithm 3.18 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F^{\mathbf{v}}_0A$-submodule V of a free A-module with shift vector $\mathbf{s}$, this algorithm computes $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}\,V$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 3.1 is satisfied, a finite set E, an $F^{\mathbf{v}}_0A$-module $V = {}_{F^{\mathbf{v}}_0A}\left\langle\overline{V'}\right\rangle \subseteq A^E$ with $V' \subseteq \mathbb{K}\langle\underline{x}\rangle^E$ finite and a shift vector $\mathbf{s} \in \mathbb{Z}^E$.

Output: A finite $\mathbf{w}[\mathbf{s}]$-homogeneous set $G \subseteq \mathbb{K}\langle\underline{x}\rangle^E$ inducing $F^{\mathbf{v}}_0\,\mathrm{Gr}^{\mathbf{w}}\,A$-generators of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}\,V \subseteq \mathbb{K}\langle\underline{x}\rangle^E \,/\, \left\langle \mathrm{lt}_{\mathbf{w}}(\langle I \cup S\rangle)^E\right\rangle$.

1: Determine a finite set $G \subseteq \mathbb{K}\langle\underline{x}\rangle^E$ satisfying $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V = \sum_{g\in G} F^{\mathbf{w}}_{\bullet-\deg_{\mathbf{w}[\mathbf{s}]}(g)}F^{\mathbf{v}}_0A\cdot\overline{g}$ by Algorithm 3.15 and Remark 3.16.
2: Set $G := \mathrm{lt}_{\mathbf{w}[\mathbf{s}]}(G)$.
3: **return** G.

Example 3.19. In the situation of Example 1.17.2 note that $F^{\mathbf{v}}_0T_X \cong T^V_X$ for the weight vector $\mathbf{v} = ((-\delta_{in})_{1\le i\le n}, (\delta_{im})_{1\le i\le m})$ on T_X. Hence $F^{\mathbf{v}}_0\,\mathrm{Gr}^{\mathbf{w}}\,T_X = \mathrm{Gr}^{\mathbf{w}}\,F^{\mathbf{v}}_0T_X = \mathrm{Gr}^{\mathbf{w}_{\mathbf{v}}}\,T^V_X$ for the weight vectors $\mathbf{w} = ((0)_{1\le i\le n}, (1)_{1\le i\le m})$ on T_X and $\mathbf{w}_{\mathbf{v}} = \mathbf{w}$ on T^V_X by Example 2.8. In particular, a PBW-reduction datum is computable for $F^{\mathbf{v}}_0\,\mathrm{Gr}^{\mathbf{w}}\,T_X$ by Example 2.26.2.

4. Interplay of weight filtrations on a module over the PBW-reduction-algebra A

The purpose of this section is to extend the methods from the previous section for free A-modules to quotients A^E/L. In general the $\mathbf{v}$-degree is unbounded and Lemma 3.7 does not apply. In many cases this problem can be solved by passing to $F^{\mathbf{v}}_d L$ for a suitable integer d.

Let $A = (\mathbb{K}\langle\underline{x}\rangle, S, I, \prec)$ be a PBW-reduction-algebra and $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ two weight vectors on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight. Given a finite set E and $L', V', W' \subseteq A^E$ finite subsets, $L := {}_A\langle L'\rangle$ and $M := A^E/L$, we consider the $F^{\mathbf{v}}_{\bullet}A$- and $F^{\mathbf{w}}_{\bullet}A$-submodules

$$V := {}_{F^{\mathbf{v}}_0 A}\left\langle \overline{V'} \right\rangle \subseteq M \quad \text{and} \quad W := {}_{F^{\mathbf{w}}_0 A}\left\langle \overline{W'} \right\rangle \subseteq M,$$

respectively. For any finite set $N \subseteq A^E$ or element $a \in A^E$, we denote by $\tilde{N}$ or $\tilde{a}$ a (set of) representatives in $\mathbb{K}\langle\underline{x}\rangle^E$.

We extend our list of assumptions from Assumption 3.1 as follows:

Assumption 4.1. Assumption 3.1.1 and 2 holds if we replace A by $\operatorname{Gr}^{\mathbf{w}} A$.

Example 4.2. In the setting of Example 3.5 Assumption 4.1 holds by Example 2.26.1, Lemma 2.12, Lemma 2.13 and Example 3.19.

4.1. $F^{\mathbf{v}}_0 A$-presentations of $F^{\mathbf{v}}_0 A$-submodules of A-modules

In this subsection, we only require that $\mathbf{v}$ is a weight vector on A and that Assumption 3.11 holds. To represent V as a quotient of a free $F^{\mathbf{v}}_0 A$-module we use the surjective $F^{\mathbf{v}}_0 A$-linear map

$$\varphi : F^{\mathbf{v}}_0 A^{V'} \to V,\ (v') \mapsto \overline{v'}.$$

It induces an isomorphism of $F^{\mathbf{v}}_0 A$-modules $V \cong F^{\mathbf{v}}_0 A^{V'} / \ker(\varphi)$, where

$$\ker(\varphi) = \pi_{V'}(\operatorname{syz}_A(V', L')) \cap F^{\mathbf{v}}_0 A^{V'}.$$

The preceding intersection is computable by Algorithm 2.21. Hence we obtain:

Algorithm 4.3 Given a weight vector $\mathbf{v}$ on A and an $F^{\mathbf{v}}_0 A$-submodule V of a finitely presented A-module, this algorithm represents V as a quotient of a free $F^{\mathbf{v}}_0 A$-module.

Input: A weight vector $\mathbf{v} \in \mathbb{Z}^n$ on A such that Assumption 3.1.1 holds, a finite set E, an A-module $M := A^E / {}_A\langle L'\rangle$ and a submodule $V := {}_{F^{\mathbf{v}}_0 A}\left\langle \overline{V'} \right\rangle \subseteq M$ with $L', V' \subseteq A^E$ finite.

Output: A finite set $Q \subseteq F^{\mathbf{v}}_0 A^{V'}$ such that $F^{\mathbf{v}}_0 A^{V'} / {}_{F^{\mathbf{v}}_0 A}\langle Q\rangle \cong V$ via $\overline{a} \mapsto \overline{\sum_{v' \in V'} a_{v'} v'}$.

1: Compute an A-generating set S of $\operatorname{syz}_A(V', L')$ using Algorithm 1.31 with the setup of Lemma 1.37.
2: Compute an $F^{\mathbf{v}}_0 A$-generating set Q of ${}_A\langle \pi_{V'}(S)\rangle \cap F^{\mathbf{v}}_0 A^{V'}$ by Algorithm 2.21.
3: **return** Q.

4.2. Module membership problem for $F_0^{\mathbf{v}}A$-submodules of A-modules

In this subsection, we require that Assumption 3.1.1 and 2 is satisfied. We explain how to check for $a \in A^E$ whether $\overline{a} \in V = {}_{F_0^{\mathbf{v}}A}\left\langle \overline{V'} \right\rangle \subseteq M = A^E/L$, which is equivalent to

$$a \in {}_{F_0^{\mathbf{v}}A}\langle V' \rangle + L.$$

Since $\deg_{\mathbf{v}}(a), \deg_{\mathbf{v}}(V) \leq d := \max\{\deg_{\mathbf{v}}(\tilde{V}'), \deg_{\mathbf{v}}(\tilde{a})\}$ this, in turn, is equivalent to

$$a \in {}_{F_0^{\mathbf{v}}A}\langle V' \rangle + (L \cap F_d^{\mathbf{v}}A^E).$$

An $F_0^{\mathbf{v}}A$-generating set L'' of the above intersection can be determined by Algorithm 2.21. It remains to decide whether

$$a \in {}_{F_0^{\mathbf{v}}A}\langle V' \cup L'' \rangle.$$

This problem is solvable by Algorithm 3.10.

Algorithm 4.4 Given a weight vector $\mathbf{v}$ on A and two $F_0^{\mathbf{v}}A$-submodules V and P of a finitely presented A-module, this algorithm checks if $P \subseteq V$.

Input: A weight vector $\mathbf{v} \in \mathbb{Z}^n$ on A such that Assumption 3.1.1 and 2 holds, a finite set E, a module $M = A^E/{}_A\langle L' \rangle$ and submodules $V := {}_{F_0^{\mathbf{v}}A}\left\langle \overline{V'} \right\rangle, P := {}_{F_0^{\mathbf{v}}A}\left\langle \overline{P'} \right\rangle \subseteq M$ with $L', V', P' \subseteq A^E$ finite.

Output: `true` if $P \subseteq V$ and `false` otherwise.

1: Set $d := \max\{\deg_{\mathbf{v}}(\tilde{V}'), \deg_{\mathbf{v}}(\tilde{P}')\}$.
2: Compute a set L'' of $F_0^{\mathbf{v}}A$-generators of ${}_A\langle L' \rangle \cap F_d^{\mathbf{v}}A^E$ using Algorithm 2.21.
3: **if** $P' \subseteq {}_{F_0^{\mathbf{v}}A}\langle V' \cup L'' \rangle$ **then** ▷ Decide by Algorithm 3.10.
4: **return** `true`.
5: **return** `false`.

Remark 4.5. By Remark 3.11 the above algorithm can be extended to represent $\overline{p'} \in P'$ as an $F_0^{\mathbf{v}}A$-linear combination of the elements of $\overline{V'}$ if $p \in V$.

4.3. Intersection of $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodules of an A-module

In this subsection, we require that Assumption 3.1.1-3 is satisfied. Consider the A-module $M = A^E/L$ and its submodules $V = {}_{F_0^{\mathbf{v}}A}\left\langle \overline{V'} \right\rangle$ and $W = {}_{F_0^{\mathbf{w}}A}\left\langle \overline{W'} \right\rangle$. We explain how to compute the $F_0^{\mathbf{w}}A$-submodule

$$W \cap V \subseteq M.$$

Setting $I := {}_{F_0^{\mathbf{w}}A}\langle W' \rangle \cap \left({}_{F_0^{\mathbf{v}}A}\langle V' \rangle + L\right)$, we can rewrite

$$W \cap V = (I + L)/L \subseteq M. \tag{27}$$

Since $\deg_{\mathbf{v}}({}_{F_0^{\mathbf{w}}A}\langle W' \rangle) \leq \deg_{\mathbf{v}}(\tilde{W}') \leq d := \max\{\deg_{\mathbf{v}}(\tilde{V}'), \deg_{\mathbf{v}}(\tilde{W}')\}$ by Remark 3.6

$$I = {}_{F_0^{\mathbf{w}}A}\langle W' \rangle \cap \left({}_{F_0^{\mathbf{v}}A}\langle V' \rangle + (L \cap F_d^{\mathbf{v}}A^E)\right)$$

is an intersection of a finitely generated $F_0^{\mathbf{w}}A$-module with a finitely generated $F_0^{\mathbf{v}}A$-module. Algorithm 2.21 yields a finite set of $F_0^{\mathbf{v}}A$-generators L'' of $L \cap F_d^{\mathbf{v}}A$. Finally

$$I = {}_{F_0^{\mathbf{w}}A}\langle W' \rangle \cap {}_{F_0^{\mathbf{v}}A}\langle V' \cup L'' \rangle$$

can be computed as in Subsection 3.3.

Algorithm 4.6 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A, an $F_0^{\mathbf{v}}A$-submodule V and an $F_0^{\mathbf{w}}A$-submodule W of a finitely presented A-module, this algorithm computes $V \cap W$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 3.1.1-.3 is satisfied, a finite set E, an A-module $M := A^E/{}_A\langle L'\rangle$, submodules $V := {}_{F_0^{\mathbf{v}}A}\left\langle\overline{V'}\right\rangle$, $W := {}_{F_0^{\mathbf{w}}A}\left\langle\overline{W'}\right\rangle \subseteq M$ with $L', V', W' \subseteq A^E$ finite.

Output: A finite set $G \subseteq A^E$ such that $V \cap W = {}_{F_0^{\mathbf{w}}A}\left\langle\overline{G}\right\rangle$.

1: Set $d := \max\{\deg_{\mathbf{v}}(\tilde{V}'), \deg_{\mathbf{v}}(\tilde{W}')\}$.
2: Determine $F_0^{\mathbf{v}}A$-generators L'' of ${}_A\langle L'\rangle \cap F_d^{\mathbf{v}}A^E$ using Algorithm 2.21.
3: Compute a set of $F_0^{\mathbf{w}}A$-generators G of ${}_{F_0^{\mathbf{w}}A}\langle W'\rangle \cap {}_{F_0^{\mathbf{v}}A}\langle V' \cup L''\rangle$ by Algorithm 3.13.
4: **return** G.

The preceding approach does not allow us to reduce the computation of $F^{\mathbf{w}}[\mathbf{s}]_\bullet V = F^{\mathbf{w}}[\mathbf{s}]_\bullet M \cap V$ to the situation of Algorithm 3.15, because the $\mathbf{v}$ is in general unbounded. Up to a fixed index $k \in \mathbb{Z}$ this is possible. Based on Algorithm 3.15, one can compute a finite set $G \subseteq A^E$ and $\mathbf{t} \in \mathbb{Z}^G$ such that

$$F^{\mathbf{w}}[\mathbf{s}]_{k'} M \cap V = \sum_{g \in G} F^{\mathbf{w}}_{k'-\mathbf{t}_g} F_0^{\mathbf{v}}A \cdot \overline{g} = \sum_{g \in G} F^{\mathbf{w}}_{k'-\deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}}A \cdot \overline{g} \text{ for } k' \leq k \tag{28}$$

and

$$F^{\mathbf{w}}[\mathbf{s}]_\bullet\, {}_{F_0^{\mathbf{v}}A}\langle G\rangle = \sum_{g \in G} F^{\mathbf{w}}_{\bullet-\mathbf{t}_g} F_0^{\mathbf{v}}A \cdot g = \sum_{g \in G} F^{\mathbf{w}}_{\bullet-\deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}}A \cdot g. \tag{29}$$

Algorithm 4.7 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{v}}A$-submodule V of a finitely presented A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{w}}[\mathbf{s}]_\bullet V$ up to index k.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 3.1.1-4 holds, a finite set E, an A-module $M := A^E/{}_A\langle L'\rangle$, a submodule $V := {}_{F_0^{\mathbf{v}}A}\left\langle\overline{V'}\right\rangle \subseteq M$ with $L', V' \subseteq A^E$ finite, a shift vector $\mathbf{s} \in \mathbb{Z}^E$ and $k \in \mathbb{Z}$.

Output: A finite set $G \subseteq A^E$ and $\mathbf{t} \in \mathbb{Z}^G$ satisfying Equations (28) and (29).

1: Set $d' := \max\{\deg_{\mathbf{v}}(P^{A,\mathbf{w}}_{k-\mathbf{s}_e}) \mid e \in E\}$.$\triangleright$ $\deg_{\mathbf{v}}(F^{\mathbf{w}}[\mathbf{s}]_k A^E) \leq d'$.
2: Set $d := \max\{d', \deg_{\mathbf{v}}(\tilde{V}')\}$.
3: Determine a set of $F_0^{\mathbf{v}}A$-generators L'' of ${}_A\langle L'\rangle \cap F_d^{\mathbf{v}}A^E$ using Algorithm 2.21.
4: Compute a finite set $G \subseteq A^E$ and $\mathbf{t} \in \mathbb{Z}^G$ satisfying $F^{\mathbf{w}}[\mathbf{s}]_\bullet\, {}_{F_0^{\mathbf{v}}A}\langle V' \cup L''\rangle = \sum_{g \in G} F^{\mathbf{w}}_{\bullet-\mathbf{t}_g} F_0^{\mathbf{v}}A \cdot g$ by Algorithm 3.15.
5: **return** $G, \mathbf{t}$.

Remark 4.8. Due to Remark 3.16 the above algorithm implicitly computes a representative $\tilde{g} \in \mathbb{K}\langle \underline{x}\rangle^E$ of $g \in G$ with $\deg_{\mathbf{w}[\mathbf{s}]}(\tilde{g}) \leq \mathbf{t}_g$.

4.4. Induced $\mathbf{w}$*-weight filtrations on* $F_0^{\mathbf{v}}A$*-submodules of* A*-modules*

In this subsection, we require that Assumption 3.1 and Assumption 4.1 are satisfied. Recall that $V = {}_{F_0^{\mathbf{v}}A}\left\langle\overline{V'}\right\rangle$ is an $F_0^{\mathbf{v}}A$-submodule of $M = A^E/L$ with $L = {}_A\langle L'\rangle$ and $\mathbf{s} \in \mathbb{Z}^E$ a shift vector. As an additional hypothesis satisfied in applications to mixed Hodge modules we require that $F^{\mathbf{w}}[\mathbf{s}]_\bullet V$ is $F^{\mathbf{w}}_\bullet F_0^{\mathbf{v}}A$-finitely generated.

Our approach is based on a general result on induced filtrations: Let $F_\bullet R$ be a filtered $\mathbb{K}$-algebra and $T \subseteq R$ a subalgebra with induced filtration. Consider an $F_\bullet R$-module $F_\bullet N$, an R-submodule $P \subseteq N$ and an T-submodule $U \subseteq N$. The filtration $F_\bullet N$ induces two $F_\bullet T$-filtrations on $Q := (U + P)/P$ as follows:

$$\begin{array}{ccc} & F_\bullet N & \\ \swarrow^{\text{subm}}_{\text{filt}} & & \searrow^{\text{quot}}_{\text{filt}} \\ F_\bullet U & & F_\bullet(N/P) \\ \text{quot}\downarrow\text{filt} & & \text{subm}\downarrow\text{filt} \\ F_\bullet^{q(U)} Q := (F_\bullet U + P)/P & \hookrightarrow & F_\bullet^s Q := F_\bullet(N/P) \cap Q. \end{array}$$

One easily sees that $F_\bullet^{q(U)} Q \subseteq F_\bullet^s Q$ and that $F_\bullet^{q(U)} Q$ depends on U, while $F_\bullet^s Q$ does not. Equality of the two filtrations can be described in of associated graded modules:

Proposition 4.9. *We have* $F_\bullet^{q(U)} Q = F_\bullet^s Q$ *if and only if*

$$\operatorname{Gr}^F(U \cap P) = \operatorname{Gr}^F U \cap \operatorname{Gr}^F P$$

under the natural identification of the above modules with submodules of $\operatorname{Gr}^F N$.

Proof. For both equalities the inclusion of the left in the right hand side holds trivially.

Assume that $F_\bullet^{q(U)} Q = F_\bullet^s Q$ and let $0 \neq n \in \operatorname{Gr}_k^F U \cap \operatorname{Gr}_k^F P$ for $k \in \mathbb{Z}$. Then there exist $u \in F_k U$ and $p \in F_k P$ such that $n = u + F_{k-1} N = p + F_{k-1} N$. This implies $u - p \in F_{k-1} N$ and thus $\overline{u} \in Q \cap F_{k-1}(N/P) = F_{k-1}^s Q = F_{k-1}^{q(U)} Q$. Hence there is some $u' \in F_{k-1} U$ and $p' \in P$ such that $u = u' + p'$. We conclude that $p' \in P \cap U$ and $p' + F_{k-1} N = u - u' + F_{k-1} N = n$ showing the first implication.

Conversely, assume $\operatorname{Gr}^F(U \cap P) = \operatorname{Gr}^F U \cap \operatorname{Gr}^F P$ and consider $q \in U + P$ with $0 \neq \overline{q} \in F_k^s Q$ for $k \in \mathbb{Z}$. By construction of $F_\bullet^s Q$, there exists $u \in U, p \in P$ such that $\overline{q} = \overline{u}$ and $u + p \in F_k N$. If $u \in F_k N$, we are done. Otherwise $p \notin F_k N$ and there is some $j > k$ such that $u + F_{j-1} N = -p + F_{j-1} N \in \operatorname{Gr}_j^F U \cap \operatorname{Gr}_j^F P = \operatorname{Gr}_j^F(U \cap P)$. Hence there exist $n \in U \cap P$, $u' \in F_{j-1} U$ and $p' \in F_{j-1} P$ such that $u = n + u'$ and $p = -n + p'$. Then $u' + p' = u + p \in F_k N$, $\overline{q} = \overline{u'}$ and $u' \in F_{j-1} N$. Iterating this argument finishes the proof. $\square$

In the following we construct an increasing sequence of finitely generated $F_0^{\mathbf{v}} A$-modules $V_k \subseteq {}_{F_0^{\mathbf{v}} A}\langle V' \rangle + L = V_k + L$ such that

$$(F^{\mathbf{w}}[\mathbf{s}]_\bullet V_k + L)/L = F^{\mathbf{w}}[\mathbf{s}]_\bullet^{q(V_k)} V \subseteq F^{\mathbf{w}}[\mathbf{s}]_\bullet^s V = F^{\mathbf{w}}[\mathbf{s}]_\bullet V$$

becomes an equality for large k. By assumption $F^{\mathbf{w}}[\mathbf{s}]_k V$ contains $F_\bullet^{\mathbf{w}} F_0^{\mathbf{v}} A$-generators of $F^{\mathbf{w}}[\mathbf{s}]_\bullet V$ for large k. For fixed $k \in \mathbb{Z}$ Algorithm 4.7 computes a set $V_k' \subseteq A^E$ such that

$$F^{\mathbf{w}}[\mathbf{s}]_{k'} V = \sum_{v \in V_k'} F^{\mathbf{w}}_{k' - \deg_{\mathbf{w}[\mathbf{s}]}(v)} F_0^{\mathbf{v}} A \cdot \overline{v} \tag{30}$$

for $k' \leq k$ and

$$F^{\mathbf{w}}[\mathbf{s}]_\bullet \, {}_{F_0^{\mathbf{v}} A}\left\langle V_k' \right\rangle = \sum_{v \in V_k'} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(v)} F_0^{\mathbf{v}} A \cdot v. \tag{31}$$

We consider only k such that $F^{\mathbf{w}}[\mathbf{s}]_k V$ is a set of $F_0^{\mathbf{v}}A$-generators of V. It suffices to take $k \geq \deg_{\mathbf{w}[\mathbf{s}]}(\tilde{V}')$. For any such $k \in \mathbb{Z}$ set $V_k = {}_{F_0^{\mathbf{v}}A}\left\langle V'_k \right\rangle$. Then

$$F^{\mathbf{w}}[\mathbf{s}]^{q(V_k)}_{\bullet} V = \sum_{v \in V'_k} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(v)} F_0^{\mathbf{v}}A \cdot \overline{v}$$

is an exhaustive filtration.

In this situation Proposition 4.9 reads:

Corollary 4.10. *We have*

$$F^{\mathbf{w}}[\mathbf{s}]_{\bullet} V = \sum_{v \in V'_k} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(v)} F_0^{\mathbf{v}}A \cdot \overline{v} \tag{32}$$

if and only if

$$\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(V_k) \cap \mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(L) = \mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(V_k \cap L). \tag{33}$$

A PBW-reduction datum of $\mathrm{Gr}^{\mathbf{w}} A$ is computable by Algorithm 2.27 due to Assumption 3.1.6, and Assumption 4.1. Algorithm 3.18 and Algorithm 2.28 compute $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}}A$-generators of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(V_k) \subseteq (\mathrm{Gr}^{\mathbf{w}}A)^E$ and $\mathrm{Gr}^{\mathbf{w}}A$-generators of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(L) \subseteq (\mathrm{Gr}^{\mathbf{w}}A)^E$, respectively. The two modules can be intersected by Algorithm 3.13 (see Remark 3.4). In the same way we compute $F_0^{\mathbf{v}}A$-generators of $V_k \cap L$ and Algorithm 3.18 yields $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(V_k \cap L)$. Finally Equation (33) can be verified using Algorithm 3.10.

This leads to the following algorithm:

Algorithm 4.11 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A, an A-submodule L and an $F_0^{\mathbf{v}}A$-submodule V of a free A-module with shift vector $\mathbf{s}$, this algorithm checks whether the quotient and the submodule filtration induced by $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}$ on $(V + L)/L$ agree.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 3.1 and Assumption 4.1 are satisfied, a finite set E, submodules $L = {}_A\langle L' \rangle$ and $V = {}_{F_0^{\mathbf{v}}A}\langle V' \rangle \subseteq A^E$ with $L', V' \subseteq A^E$ finite and a shift vector $\mathbf{s} \in \mathbb{Z}^E$.

Output: `true` if $F^{\mathbf{w}}[\mathbf{s}]^{s}_{\bullet}(V + L/L) = F^{\mathbf{w}}[\mathbf{s}]^{q(V)}_{\bullet}(V + L/L)$ and `false` otherwise.

Compute a PBW-reduction datum of $\mathrm{Gr}^{\mathbf{w}} A$ using Algorithm 2.27.
Find $\mathrm{Gr}^{\mathbf{w}}A$-generators L'' of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(L)$ by Algorithm 2.28.
Compute $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}}A$-generators V'' of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(V)$ using Algorithm 3.18.
Find $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}}A$-generators J of the intersection ${}_{F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A}\langle V'' \rangle \cap {}_{\mathrm{Gr}^{\mathbf{w}} A}\langle L'' \rangle$ using Algorithm 3.13 and Remark 3.4.
Compute $F_0^{\mathbf{v}}A$-generators K of $L \cap V$ by Algorithm 3.13 and Remark 3.4.
Determine $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}}A$-generators K' of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}({}_{F_0^{\mathbf{v}}A}\langle K \rangle)$ using Algorithm 3.18.
if $J \subseteq {}_{F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A}\langle K' \rangle$ **then** ▹ Check by Algorithm 3.10.
    return `true`.
return `false`.

Finally we obtain an algorithm to compute $F^{\mathbf{w}}[\mathbf{s}]_{\bullet} V$:

Algorithm 4.12 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{v}}A$-submodule V of a finitely presented A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{w}}[\mathbf{s}]_{\bullet} V$ if this filtration has a finite set of generators.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 3.1 and Assumption 4.1 are satisfied, a finite set E, an A-module $M := A^E/L$ with $L = {}_A\langle L' \rangle$, a submodule $V = {}_{F_0^{\mathbf{v}}A}\left\langle \overline{V'} \right\rangle \subseteq M$ with $L', V' \subseteq A^E$ finite and a shift vector $\mathbf{s} \in \mathbb{Z}^E$.

Output: A finite set $G \subseteq A^E$ and $\mathbf{t} \in \mathbb{Z}^G$ such that

$$F^{\mathbf{w}}[\mathbf{s}]_{\bullet} V = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \mathbf{t}_g} F^{\mathbf{v}}_0 A \cdot \overline{g} = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F^{\mathbf{v}}_0 A \cdot \overline{g}.$$

Set $k = \deg_{\mathbf{w}[\mathbf{s}]}(\tilde{V}')$.
Initialize an empty set $G \subseteq A^E$ and a dynamic vector $\mathbf{t} \in \mathbb{Z}^G$.
while $F^{\mathbf{w}}[\mathbf{s}]_{\bullet} V \neq \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \mathbf{t}_g} F^{\mathbf{v}}_0 A \cdot g$ **do** ▹ Test by Algorithm 4.11.
    Compute a finite set $G' \subseteq A^E$ and $\mathbf{t}' \in \mathbb{Z}^{G'}$ with $F^{\mathbf{w}}[\mathbf{s}]_{k'} V = \sum_{g \in G'} F^{\mathbf{w}}_{k' - \mathbf{t}'_g} F^{\mathbf{v}}_0 A \cdot \overline{g} = \sum_{g \in G'} F^{\mathbf{w}}_{k' - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F^{\mathbf{v}}_0 A \cdot \overline{g}$ for $k' \leq k$ using Algorithm 4.7.
    Replace G by G' and $\mathbf{t}$ by $\mathbf{t}'$.
    Increase k.
return $G, \mathbf{t}$.

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