THE DIFFERENTIAL STRUCTURE OF THE BRIESKORN LATTICE

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Abstract. The Brieskorn lattice H'' of an isolated hypersurface singularity with Milnor number μ is a free $\mathbb{C}\{\{s\}\}$-module of rank μ with a differential operator $t = s^2\partial_s$. Based on the mixed Hodge structure on the cohomology of the Milnor fibre, M. Saito constructed $\mathbb{C}\{\{s\}\}$-bases of H'' for which the matrix of t has the form $A = A_0 + A_1 s$. We describe an algorithm to compute the matrices A_0 and A_1. They determine the differential structure of the Brieskorn lattice, the spectral pairs and Hodge numbers, and the complex monodromy of the singularity.

1. The Milnor Fibration

Let $f : (\mathbb{C}^{n+1}, \underline{0}) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated critical point and Milnor number $\mu = \dim_{\mathbb{C}} \mathbb{C}\{\underline{x}\}/\langle \underline{\partial}(f)\rangle$ where $\underline{x} = x_0, \dots, x_n$ is a complex coordinate system of $(\mathbb{C}^{n+1}, 0)$ and $\underline{\partial} = \partial_{x_0}, \dots, \partial_{x_n}$. By the finite determinacy theorem, we may assume that $f \in \mathbb{C}[\underline{x}]$. By E.J.N Looijenga [7, 2.B], for a good representative $f : X \longrightarrow T$ where $T \subset \mathbb{C}$ is an open disk at the origin, the restriction $f : X' \longrightarrow T'$ to $T' = T\backslash\{0\}$ and $X' = X\backslash f^{-1}(0)$ is a $\mathscr{C}^\infty$ fibre bundle unique up to diffeomorphism, the Milnor fibration. By J. Milnor [9, 6.5], the general fibre $X_t = f^{-1}(t)$, $t \in T'$, is homotopy equivalent to a bouquet of μ n-spheres and, in particular, its reduced cohomology is $\widetilde{\mathrm{H}}^k(X_t) \cong \delta_{k,n}\mathbb{Z}^\mu$ where δ is the Kronecker symbol. Since T' is locally contractible, the n-th cohomologies $\mathrm{H}(U) = \mathrm{H}^n(X_U)$ of $X_U = f^{-1}(U)$ form a locally free $\mathbb{Z}$-sheaf of rank μ and $\mathrm{H}_{\mathbb{C}} = \mathrm{H} \otimes_{\mathbb{Z}} \mathbb{C}$ is a complex local system of dimension μ. Hence, the sheaf of holomorphic sections $\mathscr{H} = \mathrm{H} \otimes_{\mathbb{Z}} \mathscr{O}_{T'}$ of $\mathrm{H}_{\mathbb{C}}$ is a locally free $\mathscr{O}_{T'}$-sheaf of rank μ, the cohomology bundle. By P. Deligne [4, 2.23], there is a natural flat connection $\nabla : \mathscr{H} \longrightarrow \mathscr{H} \otimes_{\mathscr{O}_{T'}} \Omega^1_{T'}$ on $\mathscr{H}$ with sheaf of flat sections $\mathrm{H} = \ker(\nabla)$, the Gauss-Manin connection.

2. The Monodromy Representation

Let t be a complex coordinate of $T \subset \mathbb{C}$, $i : T' \longrightarrow T$ the canonical inclusion, and $u : T^\infty \longrightarrow T'$ the universal covering of T' defined by $u(\tau) = \exp(2\pi \mathrm{i}\tau)$ for a complex coordinate τ of $T^\infty \subset \mathbb{C}$. Then the covariant derivative ∇_{∂_t} of ∇ along ∂_t induces a differential operator ∂_t on $i_*\mathscr{H}$ and the pullback $f^\infty : X^\infty = X' \times_{T'} T^\infty \longrightarrow T^\infty$

is a $\mathscr{C}^\infty$ fibre bundle with $X^\infty_\tau = X_{u(\tau)}$, the (canonical) Milnor fibre. Since T^∞ is contractible, the n-th cohomologies $H(U) = \mathrm{H}^n(X^\infty_U)$ of $X^\infty_U = (f^\infty)^{-1}(U)$ form a free $\mathbb{Z}$-sheaf of rank μ and u_*H is the sheaf of multivalued sections of H. Lifting closed paths in T' along sections of H defines the monodromy representation $\pi_1(T', t) \longrightarrow \mathrm{Aut}(\mathrm{H}_t)$ on H_t inducing the monodromy representation $\pi_1(T') \longrightarrow \mathrm{Aut}(H)$ on the cohomology H of the Milnor fibre. The image M of the counterclockwise generator of $\pi_1(T')$ is called the monodromy operator and fulfills $\mathrm{M}(s)(\tau) = s(\tau + 1)$ for $s \in H$. The sheaf H is determined by the monodromy representation up to isomorphism. The following well known theorem is due to E. Brieskorn [2, 0.6] and others.

Theorem 1 (Monodromy Theorem)**.** *The eigenvalues of the monodromy are roots of unity and its Jordan blocks have size at most* $(n+1) \times (n+1)$ *and size at most* $n \times n$ *for eigenvalue* 1*.*

3. The Gauss-Manin Connection

Let $\mathrm{M} = \mathrm{M}_s \mathrm{M}_u$ be the decomposition of M into semisimple part M_s and unipotent part M_u and let $\mathrm{N} = -\frac{\log \mathrm{M}_u}{2\pi\mathrm{i}}$ be the nilpotent part of M. Note that $-2\pi\mathrm{i}\mathrm{N} \in \mathrm{End}_\mathbb{Q}(H_\mathbb{Q})$ where $H_\mathbb{Q} = H \otimes_\mathbb{Z} \mathbb{Q}$. Let $H_\mathbb{C} = \bigoplus_\lambda H^\lambda_\mathbb{C}$ be the decomposition of $H_\mathbb{C} = H \otimes_\mathbb{Z} \mathbb{C}$ into generalized λ-eigenspaces $H^\lambda_\mathbb{C}$ of M and $\mathrm{M}^\lambda = \mathrm{M}|_{H^\lambda_\mathbb{C}}$. Note that $H_\mathbb{Q} = H^1_\mathbb{Q} \oplus H^{\neq 1}_\mathbb{Q}$ where $H^1_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C} = H^1_\mathbb{C}$ and $H^{\neq 1}_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C} = \bigoplus_{\lambda \neq 1} H^\lambda_\mathbb{C}$. Then there is an inclusion

$$H^{\mathrm{e}^{-2\pi\mathrm{i}\alpha}}_\mathbb{C} \xrightarrow{\psi_\alpha} (i_*\mathscr{H})_0$$

defined by $\psi_\alpha(A) = t^{\alpha+\mathrm{N}} A = t^\alpha \exp(N \log(t))$ with image $C^\alpha = \mathrm{im}(\psi_\alpha)$. In particular, the operators M and N act on C^α. The following lemma is an immediate consequence of the definition of ψ_α.

Lemma 2.

(1) $t \circ \psi_\alpha = \psi_{\alpha+1}$ *and* $\partial_t \circ \psi_\alpha = \psi_{\alpha-1} \circ (\alpha + \mathrm{N})$.
(2) $t : C^\alpha \longrightarrow C^{\alpha+1}$ *is bijective and* $\partial_t : C^\alpha \longrightarrow C^{\alpha-1}$ *is bijective if* $\alpha \neq 0$.
(3) *On* C^α, $t\partial_t - \alpha = \mathrm{N}$ *and* $\exp(-2\pi\mathrm{i}t\partial_t) = \mathrm{M}^{\mathrm{e}^{-2\pi\mathrm{i}\alpha}}$.
(4) $C^\alpha = \ker(t\partial_t - \alpha)^{n+1}$.

Definition 3. We call $G = \bigoplus_{-1<\alpha\leq 0} \mathbb{C}\{t\}[t^{-1}]C^\alpha \subset (i_*\mathscr{H})_0$ the local Gauss-Manin connection.

The local Gauss-Manin connection is a μ-dimensional $\mathbb{C}\{t\}[t^{-1}]$-vectorspace and a regular $\mathbb{C}\{t\}[\partial_t]$-module. The generalized α-eigenspaces C^α of the operator $t\partial_t$ define the decreasing filtration on G by free $\mathbb{C}\{t\}$-modules

$$V^\alpha = \bigoplus_{\alpha \leq \beta < \alpha+1} \mathbb{C}\{t\} C^\beta, \quad V^{>\alpha} = \bigoplus_{\alpha < \beta \leq \alpha+1} \mathbb{C}\{t\} C^\beta$$

of rank μ, the V-filtration. In contrast to the ψ_α and C^α, the V^α are independent of the coordinate t. The C^α define a splitting

$$C^\alpha \cong V^\alpha / V^{>\alpha} = \mathrm{gr}_V^\alpha G$$

of the V-filtration and we denote by lead_V the leading term with respect to this splitting. The ring $\mathbb{C}\{t\}$ is a free module of rank 1 over the ring

$$\mathbb{C}\{\{s\}\} = \Big\{ \sum_{k=0}^{\infty} a_k s^k \in \mathbb{C}[\![s]\!] \Big| \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathbb{C}\{t\} \Big\}$$

where $s = \int_0^1 \mathrm{d}t$ acts by integration. This fact is generalized by the following lemma [13, 1.3.11].

Lemma 4. *The action of $s = \partial_t^{-1}$ on $V^{>-1}$ extends to a $\mathbb{C}\{\{s\}\}$-module structure and $V^{>-1}$ is a free $\mathbb{C}\{\{s\}\}$-module of rank μ.*

Since $[\partial_t, t] = 1$, $[t, s] = s^2$ and hence

$$t = s^2 \partial_s, \quad \partial_t t = s \partial_s.$$

We call a free $\mathbb{C}\{\{s\}\}$-submodule of $V^{>-1}$ of rank μ a $\mathbb{C}\{\{s\}\}$-lattice and call a $t\partial_t$-invariant $\mathbb{C}\{\{s\}\}$-lattice saturated. A basis $\underline{e}$ of a $\mathbb{C}\{\{s\}\}$-lattice defines a matrix $A = \sum_{k \geq 0} A_k s^k$ of t by $t\underline{e} = \underline{e}A$ such that

$$t \cong A + s^2 \partial_s$$

is the basis representation of t.

4. The Brieskorn Lattice

The description of cohomology in terms of holomorphic differential forms by the de Rham isomorphism leads to the definition of the Brieskorn lattice

$$H'' = \Omega_{X,0}^{n+1} / \mathrm{d}f \wedge \mathrm{d}\Omega_{X,0}^{n-1}.$$

By E. Brieskorn [2, 1.5] and M. Sebastiani [15], the Brieskorn lattice is the stalk at 0 of a locally free $\mathscr{O}_T$-sheaf $\mathscr{H}''$ of rank μ with $\mathscr{H}''|_{T'} \cong \mathscr{H}$ and hence $H'' \subset (i_* \mathscr{H})_0$. The regularity of the Gauss-Manin connection proved by E. Brieskorn [2, 2.2] implies that $H'' \subset G$. B. Malgrange [8, 4.5] improved this result by the following theorem.

Theorem 5. $H'' \subset V^{-1}$.

By E. Brieskorn [2, 1.5], the Leray residue formula can be used to express the action of ∂_t in terms of differential forms by $\partial_t[\mathrm{d}f \wedge \omega] = [\mathrm{d}\omega]$. In particular, $sH'' \subset H''$ and

$$H''/sH'' \cong \Omega_{X,0}^{n+1} / \mathrm{d}f \wedge \Omega_{X,0}^{n} \cong \mathbb{C}\{\underline{x}\} / \langle \underline{\partial}(f) \rangle.$$

Since the $V^{>-1}$ is a $\mathbb{C}\{\{s\}\}$-module, theorem 5 implies that H'' is a free $\mathbb{C}\{\{s\}\}$-module of rank μ and the action of s can be expressed in terms of differential forms by

$$s[\mathrm{d}\omega] = [\mathrm{d}f \wedge \omega].$$

For computational purposes, we may restrict our attention to the completion of the Brieskorn lattice. E. Brieskorn [2, 3.4] proved the following theorem.

Theorem 6. *The* $\mathfrak{m}_{X,0}$- *and* $\mathfrak{m}_{T,0}$-*adic topologies on* H'' *coincide.*

While the proof of theorem 6 is highly non-trivial, the analogous statement for the $\mathbb{C}\{\{s\}\}$-structure of the Brieskorn lattice is quite elementary [13, 1.5.4].

Proposition 7. *The* $\mathfrak{m}_{X,0}$- *and* $\mathfrak{m}_{\mathbb{C}\{\{s\}\}}$-*adic topologies on* H'' *coincide.*

We call the completion $\widehat{H}''$ of H'' the formal Brieskorn lattice. Since completion is faithfully flat, $\widehat{H}''$ is a free $\mathbb{C}[\![s]\!]$-module of rank μ with a differential operator $t = s^2\partial_s$. The equality $[\underline{\partial}(f)\overline{g}\mathrm{d}\underline{x}] = s[\underline{\partial}(\overline{g})\mathrm{d}\underline{x}]$ motivates to consider the differential relation $\underline{\partial}(f) - s\underline{\partial}$. It is not difficult to prove that it defines the formal Brieskorn lattice as a quotient of $\mathbb{C}[\![s,\underline{x}]\!]$ [13, 1.5.6].

Proposition 8.

$$\mathbb{C}[\![s,\underline{x}]\!] \xrightarrow{\pi_H} \mathbb{C}[\![s,\underline{x}]\!]/\langle\underline{\partial}(f) - s\underline{\partial}\rangle\mathbb{C}[\![s,\underline{x}]\!] \cong_{\mathbb{C}[\![s]\!]} \widehat{H}''.$$

Proposition 8 is the starting point for an algorithmic approach to the local Gauss-Manin connection. Let $<_{\underline{x}}$ be a local degree ordering on $\mathbb{C}[\![\underline{x}]\!]$ such that $\deg(\underline{x}) < \underline{0}$ and $\deg(\underline{\partial}) = -\deg(\underline{x}) > \underline{0}$. One can compute a polynomial standard basis $\underline{g}$ of the Jacobian ideal $\langle\underline{\partial}(f)\rangle$ and a polynomial transformation matrix $B = \left(\overline{b}^j\right)^j$ such that $\underline{g} = \underline{\partial}(f)B$. By Nakayama's lemma, $\underline{m} = (\underline{x}^{\underline{\beta}})_{\underline{x}^{\underline{\beta}}\notin\langle\mathrm{lead}(\underline{g})\rangle}$ represents a $\mathbb{C}[\![s]\!]$-basis $[\underline{m}]$ of $\widehat{H}''$. Let $<_s$ be the local degree ordering on $\mathbb{C}[\![s]\!]$ and let $<= (<_s, <_{\underline{x}})$ be the block ordering of $<_s$ and $<_{\underline{x}}$ on $\mathbb{C}[\![s,\underline{x}]\!]$.

Definition 9.

(1) $\underline{h} = \left((g_j - s\underline{\partial}\overline{b}^j)\underline{x}^{\underline{\beta}}\right)_{j,\underline{\beta}}$.
(2) $\deg(s) = \min\deg(\underline{m}) + 2\min\deg(\underline{x}) < 0$.
(3) $N = (N_K)_{K\geq 0}$ with $N_K = K\deg(s) - 2\min\deg(\underline{x})$.
(4) $V = (V_K)_{K\geq 0}$ with $V_K = \left\{p \in \mathbb{C}[\![s,\underline{x}]\!] \,\middle|\, \deg(p) < N_K\right\} + \langle s\rangle^K \subset \mathbb{C}[\![s,\underline{x}]\!]$.

Since $\widehat{H}''$ is a free $\mathbb{C}[\![s]\!]$-module, $\underline{h}$ is a standard basis of the $\mathbb{C}[\![s]\!]$-module $\langle\underline{\partial}(f) - s\underline{\partial}\rangle\mathbb{C}[\![s,\underline{x}]\!]$. The following lemma is technical but not very deep and can be generalized to formal differential deformations [13, 2.2.10].

Lemma 10. $V = (V_K)_{K\geq 0}$ *is a basis of the* $\langle s,\underline{x}\rangle$-*adic topology of* $\mathbb{C}[\![s,\underline{x}]\!]$ *with* $\pi_H(V_K) = \langle s\rangle^K\widehat{H}''$. *If* $s^\alpha\mathrm{lead}(h_{j,\underline{\beta}}) \in V_K$ *then* $s^\alpha h_{j,\underline{\beta}} \in V_K$.

Lemma 10 leads to a normal form algorithm for the Brieskorn lattice [13, 2.2.12]. It computes a normal form with respect to $\underline{h}$ and hence the $[\underline{m}]$-basis representation in H''. The normal form computation up to a given degree can be continued up to any higher degree without additional computational effort. The normal form algorithm for the Brieskorn lattice is a special case of a modification of Buchberger's normal form algorithm [3] for power series rings where termination is replaced by adic convergence [13, 2.1.19].

5. Mixed Hodge Structure

By lemma 2, there is a $\mathbb{C}$-isomorphism

$$H_{\mathbb{C}} = \bigoplus_{-1<\alpha\leq 0} H_{\mathbb{C}}^{\mathrm{e}^{-2\pi\mathrm{i}\alpha}} \xrightarrow{\psi} \bigoplus_{-1<\alpha\leq 0} C^{\alpha} \cong V^{>-1}/sV^{>-1}$$

defined by $\psi = \bigoplus_{-1<\alpha\leq 0} \psi_\alpha$ and the monodromy M on $H_{\mathbb{C}}$ corresponds to $\exp(-2\pi\mathrm{i}t\partial_t)$ on $\bigoplus_{-1<\alpha\leq 0} C^\alpha$.

The Hodge filtration $F = (F_k)_{k\in\mathbb{Z}}$ on $V^{>-1}$ defined by J. Scherk and J.H.M. Steenbrink [14] is the increasing filtration by the free $\mathbb{C}\{\{s\}\}$-modules

$$F_k = F^{n-k} = (s^{-k}H'') \cap V^{>-1}$$

of rank μ. Via the splitting $C^\alpha \cong \mathrm{gr}_V^\alpha V^{>-1}$, the Hodge filtration induces an increasing Hodge filtration FC^α by $\mathbb{C}$-vectorspaces on C^α and, via ψ, on $H_{\mathbb{C}}$. The nilpotent operator $-2\pi\mathrm{i}\mathrm{N} \in \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$ defines an increasing weight filtration $W = (W_k)_{k\in\mathbb{Z}}$ centered at n resp. $n+1$ on $H_{\mathbb{Q}}^{\neq 1}$ resp. $H_{\mathbb{Q}}^1$.

Theorem 11. *The weight filtration W on $H_{\mathbb{Q}}$ and the Hodge filtration F on $H_{\mathbb{C}}$ define a mixed Hodge structure on the cohomology H of the Milnor fibre and the operator* N *is a morphism of mixed Hodge structures of type* $(-1,-1)$.

The mixed Hodge structure on the cohomology of the Milnor fibre was discovered by J.H.M. Steenbrink [16] and described in terms of the Brieskorn lattice by A.N. Varchenko [17].

The nilpotent operator N on C^α defines an increasing weight filtration $W = (W_k)_{k\in\mathbb{Z}}$ centered at n on C^α. By definition N commutes with ψ_α and hence

$$\psi_\alpha\big(WH_{\mathbb{C}}^{\mathrm{e}^{-2\pi\mathrm{i}\alpha}}\big) = \begin{cases} WC^\alpha, & \alpha \notin \mathbb{Z}, \\ W[-1]C^\alpha, & \alpha \in \mathbb{Z}. \end{cases}$$

The weight filtration $W = \bigoplus_{-1<\alpha\leq 0} \mathbb{C}\{\{s\}\}WC^\alpha$ on $V^{>-1}$ by free $\mathbb{C}\{\{s\}\}$-modules induces WC^α via the splitting $C^\alpha \cong \mathrm{gr}_V^\alpha V^{>-1}$.

The spectral pairs are those pairs $(\alpha, l) \in \mathbb{Q} \times \mathbb{Z}$ with positive multiplicity

$$d_l^\alpha = \dim_{\mathbb{C}} \mathrm{gr}_l^W \mathrm{gr}_V^\alpha \mathrm{gr}_0^F V^{>-1}.$$

Via the isomorphism ψ, they correspond to the Hodge numbers

$$h_\lambda^{p,l-p} = \dim_{\mathbb{C}} \mathrm{gr}_F^p \mathrm{gr}_l^W H_{\mathbb{C}}^\lambda$$

by $d_l^{\alpha+p} = h_{e^{-2\pi i\alpha}}^{n-p,l-n+p}$ for $-1 < \alpha < 0$ and $d_l^p = h_1^{n-p,l+1-n+p}$ and inherit the symmetry properties

$$d_l^\alpha = d_l^{2n-l-1-\alpha}, \quad d_l^\alpha = d_{2n-l}^{\alpha-n+l}, \quad d_l^\alpha = d_{2n-l}^{n-1-\alpha}$$

from the mixed Hodge structure. The spectral numbers are those numbers $\alpha \in \mathbb{Q}$ with positive multiplicity

$$d^\alpha = \dim_{\mathbb{C}} \mathrm{gr}_V^\alpha \mathrm{gr}_0^F V^{>-1} = \sum_{l \in \mathbb{Z}} d_l^\alpha$$

and have the symmetry property $d^\alpha = d^{n-1-\alpha}$.

6. M. Saito's Basis

By P. Deligne [5, 1.2.8], a morphism of mixed Hodge structures is strict for the Hodge filtration. In particular, by theorem 11, N is strict for the Hodge filtration on $H_{\mathbb{C}}$ and on $\mathrm{gr}_V V^{>-1}$. Hence, there is a direct sum decomposition $F_k C^\alpha = \bigoplus_{j \leq k} C^{\alpha,j}$ such that $\mathrm{N}(C^{\alpha,k}) \subset C^{\alpha,k+1}$, and $sC^{\alpha,k} \subset C^{\alpha+1,k-1}$. By definition of the Hodge filtration,

$$\mathrm{lead}_V(H'') = \sum_{\alpha \in \mathbb{Q}} \sum_{k \leq 0} \mathbb{C}\{\{s\}\} C^{\alpha,k} = \bigoplus_{\alpha \in \mathbb{Q}} \mathbb{C}\{\{s\}\} G^\alpha$$

where $G^\alpha = C^{\alpha,0}$. Let $<_{\mathbb{Q}\times\mathbb{Z}} = (>_{\mathbb{Q}}, >_{\mathbb{Z}})$ be the block ordering of $>_{\mathbb{Q}}$ and $>_{\mathbb{Z}}$ on the index set $\mathbb{Q} \times \mathbb{Z}$. Then the Hodge filtration defines a refinement of the V-filtration on $V^{>-1}$ by free $\mathbb{C}\{\{s\}\}$-modules $V^{\alpha,k} = F_k C^\alpha \oplus V^{>\alpha}$ of rank μ and the $C^{\alpha,k}$ define a splitting of this refined filtration compatible with s. We call the refinement the Hodge refinement and the splitting a Hodge splitting. The following lemma follows essentially from the fact that $\mathbb{C}\{\{s\}\}$ is a discrete valuation ring [13, 1.10.5,1.10.10].

Lemma 12. *Let H be a $\mathbb{C}\{\{s\}\}$-lattice and $C^{\alpha,k}$ a splitting of a refinement of the V-filtration compatible with s. Then a minimal standard basis of H is a $\mathbb{C}\{\{s\}\}$-basis and there is a reduced minimal standard basis of H.*

In particular, there is a reduced minimal standard basis of H'' for a Hodge splitting. The following proposition follows essentially from lemma 2.3 [13, 1.10.12].

Proposition 13. *Let $\underline{h}$ be a reduced minimal standard basis of H'' for a Hodge splitting. Then the $\underline{h}$-matrix A of t has degree 1. In particular,*

$$(H'', t) \xleftarrow{\underline{h}} (\mathbb{C}\{\{s\}\}^\mu, A_0 + A_1 s + s^2 \partial_s)$$

is an isomorphism. Moreover, A_1 is semisimple with eigenvalues the spectral numbers of f added by 1 and $\mathrm{gr}_V(A_0)$ can be identified with N.

Note that the matrices A_0 and A_1 in proposition 13 determine the differential structure of the Brieskorn lattice. M. Saito [10] first constructed a $\mathbb{C}\{\{s\}\}$-basis of H'' as in proposition 13 without calling it a reduced minimal standard basis.

7. The Algorithm

We describe an algorithm to compute A_0 and A_1 as in proposition 13 [13]. This algorithm can be simplified to compute the complex monodromy, the spectral numbers, or the spectral pairs only [13].

The normal form algorithm for the Brieskorn lattice in section 4 computes the $[\underline{m}]$-matrix $A = \sum_{k\geq 0} A_k s^k$ of t defined by $t[\underline{m}] = [f\underline{m}] = [\underline{m}]A$ up to any degree. We identify the columns of a matrix H with the generators of a submodule $\langle H\rangle \subset \mathbb{C}[\![s]\!]^\mu$ and denote by E the unit matrix. Then $\langle E\rangle$ is the $[\underline{m}]$-basis representation of $\widehat{H}''$. Hence, the following two statements hold for $\underline{h} = [\underline{m}]$ with $\kappa = 0$ and $H = E$.

- $(H_{\underline{h}})$ One can compute $\kappa \geq 0$ and a $\mu \times \mu$-matrix H with coefficients in $\mathbb{C}[s]$ of degree at most κ such that $\langle H\rangle$ is the $\underline{h}$-basis representation of $\widehat{H}''$ and $s^\kappa\langle E\rangle \subset \langle H\rangle$.
- $(A_{\underline{h}})$ One can compute the $\underline{h}$-matrix A of t up to any degree.

Step by step, we improve the $\mathbb{C}[\![s]\!]$-basis $\underline{h}$ and show that $(H_{\underline{h}})$ and $(A_{\underline{h}})$ hold. After the last step, A_0 and A_1 as in proposition 13 can be computed by a basis transformation of A to a reduced minimal standard basis of $\langle H\rangle$ up to a certain degree bound.

We call the canonical projection $\mathrm{jet}_k : \mathbb{C}[\![s]\!] \longrightarrow \bigoplus_{j=0}^k \mathbb{C}s^j$ the k-jet. Let the monomial ordering on $\mathbb{C}[\![s]\!]^\mu = \mathbb{C}[\![s]\!] \otimes_{\mathbb{C}} \mathbb{C}^\mu$ be the block ordering $<= (<_s, >_\mu)$ of the local degree ordering $<_s$ on $\mathbb{C}[\![s]\!]$ and the inverse ordering $>_\mu$ on the indices of the basis elements of $\mathbb{C}^\mu$.

7.1. **The Saturation of H''.** In this step, we show that $(H_{\underline{h}})$ and $(A_{\underline{h}})$ hold for a $\mathbb{C}[\![s]\!]$-basis $\underline{h}$ of a saturated $\mathbb{C}[\![s]\!]$-lattice.

The increasing sequence of $\mathbb{C}[\![s]\!]$-lattices defined by

$$\widehat{H}''_0 = \widehat{H}'', \quad \widehat{H}''_{k+1} = s\widehat{H}''_k + t\widehat{H}''_k \subset \widehat{H}''$$

is stationary since $\widehat{H}''$ is noetherian. Hence, the saturation $\widehat{H}''_\infty = \bigcup_{k\geq 0} \widehat{H}''_k$ of $\widehat{H}''$ is a saturated $\mathbb{C}[\![s]\!]$-lattice. The $[\underline{m}]$-basis representation $\langle H_k\rangle$ of $\widehat{H}''_k$ can be computed by

$$H_0 = Q_{-1} = E, \quad Q_k = \big(\mathrm{jet}_k(A) + s^2\partial_s\big)Q_{k-1}, \quad H_{k+1} = (sH_k|Q_k).$$

We successively compute the H_k and check in each step if $\langle Q_k\rangle \subset \langle H_k\rangle$ by a standard basis and normal form computation. If $\langle Q_k\rangle \subset \langle H_k\rangle$ then we stop the computation and set $\kappa = k$ and $H_\infty = H_\kappa$. Then $\langle H_\infty\rangle$ is the $[\underline{m}]$-basis representation of $\widehat{H}''_\infty$. We replace H_∞ by a minimal standard basis of $\langle H_\infty\rangle$. Then $\underline{h} = s^{-\kappa}\underline{h}H_\infty$ is a $\mathbb{C}[\![s]\!]$-basis of a saturated $\mathbb{C}[\![s]\!]$-lattice. By a normal form computation with respect to

H_∞ up to degree κ, we compute the $\underline{h}$-basis representation $\langle H_\infty^{-1}s^\kappa E\rangle = \langle \mathrm{jet}_\kappa(H_\infty^{-1}s^\kappa E)\rangle$ of $\widehat{H}''$. Since $\langle H_\infty\rangle \subset \langle E\rangle$, $s^\kappa\langle E\rangle \subset \langle H_\infty^{-1}s^\kappa E\rangle$. By a normal form computation with respect to H_∞ up to degree $\kappa+k$, one can compute the k-jet

$$\mathrm{jet}_k\big(H_\infty^{-1}(A-\kappa sE+s^2\partial_s)H_\infty\big) = \mathrm{jet}_k\big(H_\infty^{-1}\big(\mathrm{jet}_{\kappa+k}(A-\kappa sE)+s^2\partial_s\big)H_\infty\big)$$

of the $\underline{h}$-matrix of t for any $k\geq 0$.

7.2. **The V-Filtration.** In this step, we show that $(H_{\underline{v}})$ and $(A_{\underline{v}})$ hold for a $<_{\mathbb{Q}}$-increasingly ordered $\mathbb{C}[\![s]\!]$-basis $\underline{v}$ of a $\widehat{V}^\alpha$ compatible with the direct sum decomposition $\widehat{V}^\alpha/s\widehat{V}^\alpha \cong \bigoplus_{\alpha\leq\beta<\alpha+1} C^\beta$.

Since $\underline{h}$ is a $\mathbb{C}[\![s]\!]$-basis of a saturated $\mathbb{C}\{\{s\}\}$-lattice, $A_0=0$ and, by theorem 1, the eigenvalues of A_1 are rational. In order to compute the eigenvalues of A_1, we transform A_1 to Hessenberg form and factorize the characteristic polynomials of its blocks. Then we compute a constant $\mathbb{C}[\![s]\!]$-basis transformation such that $A_1 = \mathrm{diag}(\alpha_1,\dots,\alpha_\mu)+N$ with $\alpha_1\leq\cdots\leq\alpha_\mu$ where $\mathrm{diag}(\alpha_1,\dots,\alpha_\mu)$ denotes the diagonal matrix with entries $\alpha_1,\dots,\alpha_\mu$. If $\alpha_\mu-\alpha_1<1$ then $\underline{v}=\underline{h}$ is a $<_{\mathbb{Q}}$-increasingly ordered $\mathbb{C}[\![s]\!]$-basis $\underline{v}$ of a $\widehat{V}^\alpha$ compatible with the direct sum decomposition $\widehat{V}^\alpha/s\widehat{V}^\alpha \cong \bigoplus_{\alpha\leq\beta<\alpha+1} C^\beta$. If $\alpha_\mu-\alpha_1\geq 1$ then we proceed as follows. Let

$$A = \begin{pmatrix} A^{1,1} & A^{1,2} \\ A^{2,1} & A^{2,2}\end{pmatrix}$$

such that $A_0=0$, $A_1^{1,2}=0$, $A_1^{2,1}=0$, and the eigenvalues of $A_1^{1,1}$ are the eigenvalues α of A_1 with $\alpha<\alpha_1+1$. Then the $\mathbb{C}[\![s]\!][s^{-1}]$-basis transformation

$$H\mapsto \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 1\end{pmatrix} H,\quad A\mapsto \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 1\end{pmatrix}(A+s^2\partial_s)\begin{pmatrix} s & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} A^{1,1}+s & \frac{1}{s}A^{1,2} \\ sA^{2,1} & A^{2,2}\end{pmatrix}$$

decreases $\alpha_\mu-\alpha_1$ and the degree up to which A is computed by 1 and increases κ by 1. After at most n such transformations, $\alpha_\mu-\alpha_1<1$.

7.3. **The Canonical V-Splitting.** In this step, we show that $(H_{\underline{c}})$ and $(A_{\underline{c}})$ hold for a $<_{\mathbb{Q}}$-increasingly ordered $\mathbb{C}$-basis $\underline{c}$ of a direct sum $\bigoplus_{\alpha\leq\beta<\alpha+1} C^\beta$ compatible with the direct sum.

Let $\underline{c}$ be the image of $[\underline{v}]$ under the splitting $\widehat{V}^\alpha/s\widehat{V}^\alpha \cong \bigoplus_{\alpha\leq\beta<\alpha+1} C^\beta$. By Nakayama's lemma, $\underline{c}$ is a $\mathbb{C}$-basis of $\bigoplus_{\alpha\leq\beta<\alpha+1} C^\beta$ compatible with the direct sum. The eigenvalues of the commutator $[\cdot,A_1]\in\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\mu^2})$ are the differences of the eigenvalues of A_1. Since $\alpha_\mu-\alpha_1<1$, $[\cdot,A_1]-k\in\mathrm{GL}_{\mu^2}(\mathbb{C})$ for $k\geq 1$. Let $U=\sum_{j=0}^\infty U_j s^j$ be the $\mathbb{C}[\![s]\!]$-basis transformation defined by $\underline{c}=\underline{v}U$. Then $U_0=E$ and $UA_1s = (A+s^2\partial_s)U$ or equivalently

$$U_k = \big([\cdot,A_1]-k\big)^{-1}\sum_{j=0}^{k-1} A_{k-j+1}U_j$$

for $k \geq 1$ and hence one can compute U up to any degree. Since $U_0 = E$ and $\kappa \geq 0$, $\mathrm{jet}_\kappa(U)$ is a minimal standard basis of $\langle E \rangle$. By a normal form computation with respect to U up to degree κ, we compute the $\underline{c}$-basis representation $\langle U^{-1}H \rangle = \langle \mathrm{jet}_\kappa(\mathrm{jet}_\kappa(U)^{-1}H) \rangle$ of $\widehat{H}''$ and $A_1 s$ is the $\underline{c}$-matrix of t.

7.4. **A Hodge Splitting.**
In this step, we show that $(H_{\underline{f}})$ and $(A_{\underline{f}})$ hold for a $<_{\mathbb{Q}\times\mathbb{Z}}$-decreasingly ordered $\mathbb{C}$-basis $\underline{f}$ of a direct sum $\bigoplus_{\alpha \leq \beta < \alpha+1} \bigoplus_{k \in \mathbb{Z}} C^{\beta,k}$ compatible with the direct sum and that one can compute A_0 and A_1 as in proposition 13.

We compute a standard basis of H up to degree κ in order to compute the $\underline{c}$-basis representation of the Hodge filtration F. The nilpotent part of A_1 is the $\underline{c}$-basis representation of N. By computing images and quotients of $\mathbb{C}$-vectorspaces, we compute the $\underline{c}$-basis representation of a Hodge splitting $F_k C^\beta = \bigoplus_{j \leq k} C^{\beta,j}$. Then we compute a constant $\mathbb{C}[\![s]\!]$-basis transformation $\underline{f} = \underline{c}U$ such that $\underline{f}$ is a $<_{\mathbb{Q}\times\mathbb{Z}}$-decreasingly ordered $\mathbb{C}$-basis of the direct sum $\bigoplus_{\alpha \leq \beta < \alpha+1} \bigoplus_{k \in \mathbb{Z}} C^{\beta,k}$ compatible with the direct sum.

We replace H by a reduced minimal standard basis of $\langle H \rangle$ up to degree $\kappa + 1$. By a normal form computation with respect to H up to degree $\kappa + 1$, we compute the 1-jet

$$\mathrm{jet}_1\big(H^{-1}(A + s^2\partial_s)H\big) = \mathrm{jet}_1\big(\mathrm{jet}_{\kappa+1}(H)^{-1}\big(\mathrm{jet}_{\kappa+1}(A) + s^2\partial_s\big)\mathrm{jet}_{\kappa+1}(H)\big)$$

of the $\underline{c}H$-matrix A of t in order to compute A_0 and A_1 as in proposition 10.

8. An Example

The algorithm in section 7 is implemented in the computer algebra system Singular [6] in the procedure `tmatrix` in the library `gaussman.lib` [12]. In an example Singular session, we compute the differential structure of the Brieskorn lattice of the singularity of type $T_{2,5,5}$ defined by the polynomial $f = x^2y^2 + x^5 + y^5$.

First, we load the Singular library `gaussman.lib`:

```
> LIB "gaussman.lib";
```

Then, we define the local ring $R = \mathbb{Q}[x,y]_{\langle x,y \rangle}$ with the local degree ordering `ds` as monomial ordering and the polynomial $f = x^2y^2 + x^5 + y^5 \in R$:

```
> ring R=0,(x,y),ds;
> poly f=x2y2+x5+y5;
```

Finally, we compute A_0 and A_1 as in proposition 10:

```
> list A=tmatrix(f);
```

The result is the list `A=A[1],A[2]` such that $\texttt{A}[\texttt{i}+\texttt{1}] = A_i$ and

$$A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad A_1 = \operatorname{diag}\Big(\frac{1}{2}, \frac{7}{10}, \frac{7}{10}, \frac{9}{10}, \frac{9}{10}, 1, \frac{11}{10}, \frac{11}{10}, \frac{13}{10}, \frac{13}{10}, \frac{3}{2}\Big).$$

By proposition 10, $\big(H'', t\big) \cong \big(\mathbb{C}\{\{s\}\}^\mu, A_0 + sA_1 + s^2\partial_s\big)$ and the spectral pairs are $\big(-\frac{1}{2}, 2\big)$, $\big(-\frac{3}{10}, 1\big)^2$, $\big(-\frac{1}{10}, 1\big)^2$, $(0, 1)$, $\big(\frac{1}{10}, 1\big)^2$, $\big(\frac{3}{10}, 1\big)^2$, $\big(\frac{1}{2}, 0\big)$.

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