TROPICAL RESOLUTIONS OF CONFIGURATION HYPERSURFACES

DANIEL BATH, GRAHAM DENHAM, MATHIAS SCHULZE, AND ULI WALTHER

Abstract. Configuration polynomials generalize the Kirchhoff polynomial of a graph, as well as the Symanzik polynomials that appear in the denominators of Feynman integrands. The *configuration hypersurfaces* cut out by such polynomials are typically highly singular, which poses a challenge for the evaluation of Feynman integrals even in simplified settings.

In this paper, we provide a two-step recipe for a resolution of singularities of any irreducible configuration hypersurface. We first consider the normalization of the Nash blow-up, which we identify with an incidence variety introduced by Bloch [Blo; Blo20]. This variety is typically still not smooth, but it is the closure of a smooth subvariety of a torus. The latter then a smooth, tropical compactification, using work of Tevelev. We construct explicitly such a compactification and a morphism to the normalized Nash blow-up for every configuration, described in terms of bipermutohedral matroid combinatorics introduced by Ardila, Denham and Huh [ADH23].

Along the way, we find that the normalized Nash blow-up of the configuration hypersurface has strongly F-regular singularities in positive characteristic. We deduce this by certifying F-rationality of its biprojective cone, and infer from it that the normalized Nash blow-up has rational singularities over the complex numbers.

Contents

2020 *Mathematics Subject Classification.* Primary 14N20, 32S45; Secondary 05C31, 14T20, 81Q30.

Key words and phrases. Configuration, matroid, graph, incidence, determinantal, singularity, resolution, Feynman, Kirchhoff, Symanzik, Nash, schön, conormal fan.

DB was supported by FWO grant #1282226N.

GD was supported by NSERC of Canada.

UW was supported by NSF grant DMS-2100288 and by Simons Foundation Collaboration Grant for Mathematicians #580839 and SFI-MPS-TSM-00012928.

1. Introduction

Feynman diagrams encode particle interactions in high-energy physics, and Feynman integrals can be used to compute the probabilities of specific interactions. Even in simplified settings that neglect mass and momenta, evaluation of Feynman integrals poses serious challenges due to singularities and issues with convergence.

The Feynman integrand is the square root of a rational function, the denominator of which involves the *(first) Symanzik polynomial* $\sum_{T\in\mathcal{B}(G)}\prod_{e\notin T}x_e$, where $\mathcal{B}(G)$ denotes the set of spanning trees in the Feynman graph G. The convergence properties and evaluation techniques for the Feynman integral, then, depend on the geometry of the *graph hypersurface* cut out by the Symanzik polynomial. This hypersurface is typically highly singular, and the main focus of this paper is to provide an explicit resolution of its singularities, expressed entirely in terms of combinatorial data from the Feynman graph.

The search for a resolution of singularities for Feynman integrands was initiated by Bloch, Esnault and Kreimer [BEK06], and one of their key insights was that the construction of Symanzik polynomials generalizes naturally from graphs to realizable matroids. More precisely, they found that the first and second Symanzik polynomials are both instances of *configuration polynomials* ψ_W; these are induced by the choice of a subspace W inside a vector space V with distinguished basis—compare Proposition 2.11. The mathematical advantage of the larger world of configuration polynomials is a greater flexibility; for example there is a general notion of duality that extends that of planar graphs. Our methods and results apply to this more general setting and make use of the added features.

From now on, let $V := \bigoplus_{i\in E}\mathbb{K}\cdot x_i$ denote the vector space spanned by all edge variables x_i in the Feynman integral, and consider the Nash blow-up of the (projective) configuration hypersurface $X_W \subseteq \mathbb{P}V$. This approach was pioneered in [BEK06, §4], but it rarely provides a resolution of singularities of the configuration hypersurface.

Indeed, it follows from §3.3, 3.7 that the Nash blow-up is often not even normal, and it can be smooth only when the configuration comes from a *round matroid*. Since the class of round matroids intersects more or less vacuously with the configurations coming from interesting Feynman diagrams, one must go further.

In subsequent years, Bloch [Blo; Blo20] introduced a complete intersection variety Λ_W of bihomogeneous quadrics inside $\mathbb{P}V \times \mathbb{P}V^\star$ mapping surjectively onto X_W. We show (Theorem 3.44) that Λ_W is in fact the normalization of the Nash blow-up, giving a commutative diagram

$$\textbf{(1.1)} \qquad \begin{array}{ccccc} \mathbb{P}V \times \mathbb{P}V^\star & \xrightarrow{=} & \mathbb{P}V \times \mathbb{P}V^\star & \xrightarrow{p_2} & \mathbb{P}V^\star \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid \\ \Lambda_W & \twoheadrightarrow & \mathrm{Nash}(X_W) & \twoheadrightarrow & X_W \end{array}$$

in which the maps in the lower row are birational.

Unfortunately, the variety Λ_W generally does not provide a resolution of singularities; however, the situation has markedly improved. While X_W can have singularities on the big torus in $\mathbb{P}V$, the restriction Λ_W° of Λ_W to the big torus inside $\mathbb{P}V \times \mathbb{P}V^\star$ is always smooth (Proposition 4.4). This means that, in order to resolve the singularities of Λ_W (and hence of X_W), we can make use of the techniques of *tropical compactifications* developed by Tevelev [Tev07] and refined by Hacking [Hac08]. Roughly speaking, one blows up (torus-equivariantly) the product of projective spaces to a new toric variety $\mathbb{P}(\Delta)$ while maintaining an isomorphism on the torus. Under suitable conditions (which we show to hold in our situation), the closure $\widetilde{\Lambda}_W = \widetilde{\Lambda}_W(\Delta)$ of the smooth subvariety Λ_W° in $\mathbb{P}(\Delta)$ is guaranteed to be smooth as well, and one obtains an embedded resolution

$$\begin{array}{ccc} \mathbb{P}(\Delta) & \twoheadrightarrow & \mathbb{P}V \times \mathbb{P}V^\star \\ \cup\!\mid & & \cup\!\mid \\ \widetilde{\Lambda}_W & \twoheadrightarrow & \Lambda_W. \end{array}$$

Tevelev's theory states that the components of the boundary $\widetilde{\Lambda}_W \setminus \Lambda_W^\circ$ are restrictions of torus orbits in $\mathbb{P}(\Delta)$: as such, their incidence relations are the same as those of the torus orbits and are governed by combinatorics of the associated matroid.

Tropical compactifications of Λ_W° are not unique; rather, they are indexed by unimodular fan structures supported on its tropicalization. We show that the tropicalization $\mathrm{trop}(\Lambda_W^\circ)$ is isomorphic to the support of the product of the Bergman fans of the matroids of W and its dual $W^\perp$ (Proposition 4.11). This (linear) isomorphism μ comes from a description of Λ_W, birationally, as the graph of the Hadamard product of the projective linear spaces $\mathbb{P}W$ and $\mathbb{P}W^\perp$ (Proposition 4.4).

At this point, a combinatorialist might view the problem to be solved, since these fan structures are well-understood. The remaining subtlety, however, is that the isomorphism so constructed does not respect the product structure of (**1.1**); thus, in order to obtain a tropical compactification $\widetilde{\Lambda}_W$ together with a well-defined map to the configuration hypersurface, some additional refinement is required. For this purpose, we make use of the bipermutohedral fan $\Sigma_{E,E}$ introduced by Ardila, Denham and Huh [ADH23], with which we recall in §4.2. The *square conormal fan* $\Sigma_{-\mathsf{M},\mathsf{M}^\perp}$ is the induced subfan that refines a product of Bergman fans $(-\Sigma_{\mathsf{M}}) \times \Sigma_{\mathsf{M}^\perp}$ (Proposition 4.27). We let $\widetilde{\Delta}_{\mathsf{M}}$ denote its image under the isomorphism μ. In §4.5, we prove one of our main results:

Theorem 1.2. *If W is a configuration with connected matroid M, then the fan $\widetilde{\Delta}_{\mathsf{M}}$ gives a tropical compactification $\widetilde{\Lambda}_W$ such that $\widetilde{\Lambda}_W \to \Lambda_W \to X_W$ is a resolution of singularities, that is, a birational surjection with a smooth source.*

By the nature of a tropical compactification, the boundary structure of $\widetilde{\Lambda}_{\mathsf{M}}$ is described completely in terms of the fan $\Sigma_{-\mathsf{M},\mathsf{M}^\perp}$.

The normalized Nash blow-up Λ_W is in some sense minimal over X_W but often not smooth, whereas our tropical resolution is smooth, but often not minimal. That is, the composition above is a bijection on an open subset of X_W which is, in general, properly contained in its subscheme of regular points. As a trade-off, however, the structure of the resolution is entirely combinatorial, and its structure (in the case of graph hypersurfaces) can be read directly from the Feynman diagram.

As noted, we show that the smoothness of Λ_W is equivalent to the *roundness* of the underlying matroid, which is the condition that all complements of (proper) flats span the matroid. For graphical configurations, this means that our tropical techniques for resolving X_W by further resolving Λ_W are required in nearly all cases. Nonetheless, Λ_W always has a number of very interesting properties, both from the geometric and the algebraic viewpoint; the rest of the introduction summarizes our findings about Λ_W (and related varieties) in and of themselves.

If the matroid of W is connected, then Λ_W is a rational image of a product of projective linear spaces, via a Hadamard product construction and, in particular, is irreducible (Corollary 3.9). Additionally, as [BEK06] already observed, Λ_W is always a complete intersection in $\mathbb{P}V \times \mathbb{P}V^\star$.

In §3.8, we observe a combinatorial formula for its motivic class in $K_0(\mathrm{Var}_{\mathbb{C}})$, irrespective of roundness:

Theorem 1.3. *If M denotes the matroid of a configuration W defined over $\mathbb{C}$ and $\mathbb{L} = [\mathbb{A}^1]$ is the class of the affine line, then*

$$[\Lambda_W] = \sum_{\substack{F \in \mathscr{L}_{\mathsf{M}} \\ F \neq E}} \overline{\chi}_{\mathsf{M}/F}(\mathbb{L}) \cdot (\mathbb{L}^{|E| - \mathrm{rank}(\mathsf{M} \setminus F)} - 1)$$

in Grothendieck's ring of varieties. Here, $\overline{\chi}_{\mathsf{M}}(t)$ denotes the reduced characteristic polynomial of a matroid M and $\mathscr{L}_{\mathsf{M}}$ is the lattice of flats of M.

In particular, we find that $[\Lambda_W] \in \mathbb{Z}[\mathbb{L}]$ is an *integer* class. By contrast, $[X_W]$ generally is not an integer class but rather can be an arbitrarily complicated member of the Grothendieck ring of varieties, in a sense that is made precise in [BB03].

Prompted by the minimal model program, many singularitt classes have crystallized Y that can be tested via a resolution of singularities $\pi\colon \widetilde{Y} \to Y$ in characteristic zero. Two very important and successful classes of this type are that of *rational singularities*, requiring the identity $R\pi_*(\mathscr{O}_{\tilde{Y}}) = \mathscr{O}_Y$, and the weaker *du Bois* property. In characteristic $p > 0$, from the seminal work of Hochster and Huneke on tight closure, notions have emerged that rest on asymptotic numeric data of the behavior of (iterates of) the Frobenius endomorphism. For example, *F-purity* asks that viewing any module in characteristic $p > 0$ through the p-power map should preserve monomorphisms. This turns out to be approximately asking for the Frobenius to be a split morphism, and a strengthening is *strong F-regularity* which requires this splitting for the composition of the p-th power map (or one of its iterates) with multiplication by an arbitrary non-zerodivisor. A third property is *F-rationality*, slightly weaker than strong F-regularity and based on asymptotics of the Frobenius on local cohomology.

Astonishingly, work of Smith, Hara and others has shown that these properties in positive and zero characteristics respectively are very intimately related; see §3.6 for details of this interplay. Moreover, even though the characteristic p approach requires ostensibly the checking of infinitely many containments of ideals, in many situations it can be done successfully in practice, and a number of varieties have been certified as being du Bois or as having rational singularities by showing the positive characteristic counterparts. Our last major item in the introduction bears witness to this pattern.

Let $\hat{\Lambda}_W$ be the (affine) variety in $V \times V^\star$ defined by the equations of the complete intersection that cut out Λ_W in $\mathbb{P}V \times \mathbb{P}V^\star$. Then $\hat{\Lambda}_W$ is the union of two reduced components, $\hat{\Lambda}_W = (W \times \{0\}) \cup \hat{\Lambda}_{W,0}$, and $\hat{\Lambda}_{W,0}$ is the biprojective cone of Λ_W. In Theorem 3.31, we show:

Theorem 1.4. *The cone $\hat{\Lambda}_{W,0}$ is F-rational in every positive characteristic when the field is perfect.*[1] *Consequently,*

(1) $\hat{\Lambda}_{W,0}$ is Cohen–Macaulay and normal in all characteristics, and has rational singularities over the complex numbers.
(2) Λ_W is strongly F-regular in positive characteristic, and has all the properties listed for $\hat{\Lambda}_{W,0}$.

Moreover, we find that $\hat{\Lambda}_{W,0}$ has Cohen–Macaulay type equal to the rank of the matroid, and its coordinate ring has a very simple free resolution over the coordinate ring of $V \times V^\star$ (Theorem 3.21.)

1.1. **Outline.** In Section §2 we recall some definitions as well as basic results, old and new, on configuration polynomials. Section §3 revisits Bloch's incidence variety Λ_W. We characterize when it is smooth in §3.5, and we establish the F-rationality of the affine version in Section §3.6. We investigate the relationship between the Nash blow-up of the configuration hypersurface and Bloch's variety in §3.7, using the results of the previous section to show that Λ_W is normal. In Section §4, we develop a resolution of singularities for Bloch's variety, hence for the configuration hypersurface, by tropical methods.

[1] or just F-finite

1.2. **Assumptions and reference texts.** If the denominator in a Feynman diagram factors, one can decompose the Feynman integral into factors as well. This happens precisely when the underlying matroid is not connected. Our theorems thus focus on the connected case, although in some of our constructions we do need to pass through disconnected matroids. Similarly, the basic definitions can be made over any field, but we often require geometric scenarios. For simplicity, we will assume throughout that our base field $\mathbb{K}$ is always perfect. Further assumptions about the field or matroid are indicated at the at the start of sections/subsections when stronger hypotheses are appropriate.

Our reference for matroid theory is the book of Oxley [Oxl11]. For toric geometry we refer the reader to the book of Cox, Little and Schenck [CLS11], and for tropical geometry to that of Maclagan and Sturmfels [MS15]. For matters on the Frobenius morphism we refer to the notes by Ma and Polstra [MP22a].

1.3. **Acknowledgements.** The authors thank the Centro Ennio de Giorgi for its hospitality, where this project started. The first three authors also thank the Max Planck Institute for Mathematics (Bonn) for its support during its development.

2. Matroids, configurations, and polynomials

Our starting point is the *configuration polynomial* ψ_W, which was introduced by Bloch, Esnault and Kreimer [BEK06] and includes the Symanzik and Kirchhoff polynomials as special cases. We recall its definition and establish our notation (see [Den+22, §2] for further details).

2.1. **Configurations.** We fix a base field $\mathbb{K}$ and a finite set $E = \{1, \dots, n\}$. We denote by

$$V := \mathbb{K}^E := \bigoplus_{i \in E} \mathbb{K} \cdot x_i$$

the n-dimensional $\mathbb{K}$-vector space with basis $x = x_1, \dots, x_n$. For any subset $F \subseteq E$ the vector space $\mathbb{K}^F$ with basis $x_F = (x_i)_{i \in F}$ can be realized either as a sub- or a quotient space of V. The assignment $F \mapsto \mathbb{K}^F$ defines two functors from the power set of E, covariantly to the category of monomorphisms, and contravariantly to that of epimorphisms of $\mathbb{K}$-vector spaces. In particular, we denote the coordinate projection associated to $F \subseteq E$ by

$$\pi_F \colon \mathbb{K}^E \twoheadrightarrow \mathbb{K}^F.$$

A *configuration* is a $\mathbb{K}$-linear subspace $W \subseteq V$. We note the inclusion map by

$$\ell = (\ell_1, \dots, \ell_n) \colon W \hookrightarrow V.$$

Then W is a $\mathbb{K}$-linear realization of an underlying *matroid* $\mathsf{M} := \mathsf{M}_W$ with set of bases

$$\mathcal{B}_{\mathsf{M}} := \left\{ B \subseteq E \mid \pi_B \text{ restricts to an isomorphism } W \to \mathbb{K}^B \right\}$$

In this way, any matroid property of M_W can be considered as a property of a configuration. In particular, we refer to W as *connected* if M_W is so. We often abbreviate the *rank* of W by

$$r := \dim_{\mathbb{K}} W = \operatorname{rank} \mathsf{M}_W.$$

In terms of a basis $w^1, \dots, w^r$ of W, the map ℓ is given by the transpose $A^\intercal$ of the $r \times n$-matrix

$$A = (w^i_j) \in \mathbb{K}^{r\times n}, \quad w^i_j := \ell_j(w^i). \tag{2.1}$$

The bases of M_W index the maximal subsets of linearly independent columns of A. For $F \subseteq E$, the matroid *deletion* $\mathsf{M}_W \backslash F$ and *contraction* M_W / F are realized by the corresponding configurations

$$W\backslash F := \pi_{E\backslash F}(W) \subseteq \mathbb{K}^{E\backslash F} \quad \text{and} \quad W/F := W \cap \mathbb{K}^{E\backslash F} \subseteq \mathbb{K}^{E\backslash F}.$$

Denote by $V^\star := \operatorname{Hom}_\mathbb{K}(V, \mathbb{K})$ the dual $\mathbb{K}$-vector space with distinguished basis $y = y_1, \dots, y_n$ dual to $x = x_1, \dots, x_n$. For $\beta \in V^\star$, we write

$$\beta_i := y_i(\beta) = \beta(x_i), \quad \beta_F := \beta|_{\mathbb{K}^F}$$

for $i \in E$ and $F \subseteq E$. Note that ℓ_i is the restriction of y_i to W. The *dual configuration* to W is the subspace

$$W^\perp := (V/W)^\star \subseteq V^\star = \bigoplus_{i\in E} \mathbb{K} \cdot y_i$$

with inclusion map $\ell^\perp = (\ell^\perp_1, \dots, \ell^\perp_n)$ realizing the dual matroid $\mathsf{M}_{W^\perp} = (\mathsf{M}_W)^\perp$. Using duality $V \cong V^{\star\star}$ we may view $x_1, \dots, x_n$ as a basis of $V^{\star\star}$ and $\ell^\perp_i$ as the restriction of x_i to $W^\perp$.

Finally, for vectors $v \in V$ (or $V^\star$, or their projectivizations), let

$$F(v) := \{i \in E \mid v_i = 0\}. \tag{2.2}$$

For a configuration W, the collection of subsets $\mathscr{L}_\mathsf{M} := \{F(w) \mid w \in W\}$ is the set of *flats* of the matroid M_W.

2.2. **Round matroids.** The following type of matroid considered by Geelen, Gerards and Whittle (see [GGW03]) will play a crucial role.

Definition 2.3. A matroid M is said to be *round* if every cocircuit (a minimal set that meets every basis of M) spans M. ♢

Proposition 2.4. *A matroid* M *on* E *is round if and only if* $\operatorname{rank}(\mathsf{M}\backslash F) = \operatorname{rank} \mathsf{M}$ *for all proper flats* F*. Equivalently, there is no partition* $E = E_1 \sqcup E_2$ *for which neither* E_1 *nor* E_2 *span. In particular, round matroids are connected.*

Proof. The maximal proper flats are precisely the complements of the cocircuits, since no basis can be contained in a proper flat (see [Oxl11, 2.1.6.(iii)]). Any flat F is an intersection of maximal proper flats; see [Oxl11, Prop. 1.7.8]. The complement $E\backslash F$ of F is thus a union of cocircuits.

If M is round and F proper, it follows that $E\backslash F$ spans and $\operatorname{rank}_\mathsf{M}(E\backslash F) = \operatorname{rank}_\mathsf{M} E$. Conversely, let C be a cocircuit: then C is the complement of a maximal proper flat H, so the rank condition implies that C spans in M.

This proves the equivalence and the remaining claims are obvious. □

Remark 2.5. A non-trivial partition as in Proposition 2.4 is "nontrivial on loops" in the terminology of Bloch (see [Blo20, §4]). Based on this partition interpretation of roundness, Kung [Kun86] calls such matroids non-split. Borissova [Bor16] characterized regular round matroids as just those of complete graphs K_n for $n \geq 2$, together with the dual matroid of the complete bipartite graph $K_{3,3}$. ♢

Example 2.6. Since a flat of rank $r-1$ has at least $r-1$ elements, if M is round, we see it is necessary to have $n \geq 2r-1$. The proper flats of the uniform matroid $U_{r,n}$ are subsets of $[n]$ of size at most $r-1$, so for uniform matroids, $n \geq 2r-1$ is also a sufficient condition to be round. ♢

2.3. **Tori and arrangements.** Each coordinate function y_i on V or x_i on $V^\star$ defines a coordinate hyperplane $\mathrm{V}(y_i)$ or $\mathrm{V}(x_i)$. The intersection of their respective complements are the *coordinate tori*

$$\hat{\mathbb{T}}_E \coloneqq V \setminus \mathrm{V}(y_1 \cdots y_n) \quad \text{and} \quad \hat{\mathbb{T}}_{E^\star} \coloneqq V^\star \setminus \mathrm{V}(x_1 \cdots x_n).$$

We let $\mathbb{T}_E$ and $\mathbb{T}_{E^\star}$ denote their respective images in $\mathbb{P}V$ and $\mathbb{P}V^\star$. Note that $i \in E$ is not a loop on M_W if and only if $\ell_i \neq 0$. In this case, $\mathrm{V}(\ell_i) = W \cap \mathrm{V}(y_i)$ is a hyperplane in W. Together they form the *hyperplane arrangements*

$$\mathscr{A}_W \coloneqq \{\mathrm{V}(\ell_i) \mid i \in E, \ell_i \neq 0\} \quad \text{and} \quad \mathscr{A}_{W^\perp} \coloneqq \left\{\mathrm{V}(\ell_i^\perp) \mid i \in E, \ell_i^\perp \neq 0\right\}$$

associated with the configurations W and $W^\perp$, respectively. Their respective complements in W and $W^\perp$ are the intersections

$$W^\circ \coloneqq W \cap \hat{\mathbb{T}}_E \quad \text{and} \quad (W^\perp)^\circ \coloneqq W^\perp \cap \hat{\mathbb{T}}_{E^\star}.$$

Their projective counterparts are

$$(\mathbb{P}W)^\circ \coloneqq \mathbb{P}(W^\circ) = (\mathbb{P}W) \cap \mathbb{T}_E \quad \text{and} \quad (\mathbb{P}W^\perp)^\circ \coloneqq \mathbb{P}((W^\perp)^\circ) = \mathbb{P}(W^\perp) \cap \mathbb{T}_{E^\star}.$$

The torus $\hat{\mathbb{T}}_E$ acts on V by linear transformations and thus on $V^\star$ by the *contragredient action*. For $t \in \hat{\mathbb{T}}_E$, $\beta \in V^\star$ and $v \in V$ it is given by

$$(t\beta)(v) = \beta(t^{-1}v) \tag{2.7}$$

and restricts to an action on $\hat{\mathbb{T}}_{E^\star}$.

2.4. **Bilinear forms.** The multiplication map $\hat{\mathbb{T}}_E \times \hat{\mathbb{T}}_E \to \hat{\mathbb{T}}_E$ extends to a $\mathbb{K}$-bilinear $\hat{\mathbb{T}}_E$-biequivariant map, the *Hadamard product*

$$Q_E \colon V \times V \to V, \quad (u,v) = \left(\sum_{i\in E} u_i x_i, \sum_{i \in E} v_i x_i\right) \mapsto \sum_{i \in E} u_i v_i x_i. \tag{2.8}$$

Considering the first argument as a parameter, dualizing yields a bilinear map

$$Q_E^\star \colon V \times V^\star \to V^\star, \quad (u,\beta) \mapsto (v \mapsto \beta \circ Q_E(u,v)).$$

Lemma 2.9. *Restricted to $\hat{\mathbb{T}}_E$ in the first argument, $Q_E^\star$ becomes the inverse contragredient action, that is, $Q_E^\star(u,\beta) = u^{-1}\beta$ for $u \in \hat{\mathbb{T}}_E$ and $\beta \in V^\star$.*

Proof. For $u \in \hat{\mathbb{T}}_E$, $\beta \in V^\star$ and $v \in V$, we have

$$Q_E^\star(u,\beta)(v) = \beta \circ Q_E(u,v) = \beta(uv) = (u^{-1}\beta)(v).$$ □

Composition of Q_E and $Q_E^\star$ with $W \hookrightarrow V$ and $V^\star \twoheadrightarrow W^\star$ leads to bilinear maps

$$Q_W \colon W \times W \to V, \quad Q_W^\star \colon W \times V^\star \to W^\star.$$

Considering the first argument as a parameter, $Q_W^\star$ becomes the dual of Q_W, that is,

$$Q_W^\star(w, \beta) = \beta \circ Q_W(w, -).$$

We use square brackets $[-]$ to denote matrices of (bi)linear maps with respect to the coordinate basis of V, a chosen basis of W and their respective dual bases. Then $[\ell] = A^\intercal$, $[Q_E] = \operatorname{diag}(x_1, \dots, x_n)$, $[Q_E^\star(-, \beta)] = \operatorname{diag}([\beta])$ and

$$[Q_W] = A[Q_E]A^\intercal, \quad \beta([Q_W]) = [\beta \circ Q_W] = [Q_W^\star(-, \beta)] = A[Q_E^\star(-, \beta)]A^\intercal. \tag{2.10}$$

2.5. **Configuration polynomials.** Consider a configuration $W \subseteq V = \mathbb{K}^E$ of rank r. Its *configuration polynomial* is the homogeneous polynomial of degree r defined (up to a nonzero square in $\mathbb{K}$ by using a basis of W for the determinant) as

$$\psi_W \coloneqq \det(Q_W) \in \operatorname{Sym}(V) = \mathbb{K}[x_1, \dots, x_n].$$

To avoid triviality, we only consider configuration polynomials for which $n > r > 0$.

Proposition 2.11 ([BEK06, Cor. 1.4])**.** *The configuration polynomial of a configuration $W \subseteq V$ is squarefree and can be written as*

$$\psi_W = \sum_{B \in \mathcal{B}_{\mathsf{M}_W}} \det(\pi_B)^2 \prod_{i \in B} x_i.$$

Remark 2.12. The first and second Symanzik polynomials of a graph are configuration polynomials (see [BEK06, §2] and [Pat10, Def. 3.6]), cf. Example 3.15. Moreover, the following conditions are equivalent: a) W is contained in the coordinate hyperplane $\mathrm{V}(x_i = 0)$; b) ψ_W does not involve x_i explicitly; c) i is a loop of M_W; d) $\ell_i = 0$. $\diamondsuit$

The configuration polynomial defines an affine/projective *configuration hypersurface*

$$\hat{X}_W \coloneqq \mathrm{V}(\psi_W) = \{\beta \in V^\star \mid \operatorname{rank}(\beta \circ Q_W) < r\} \subseteq V^\star, \quad X_W \coloneqq \mathbb{P}\hat{X}_W \subseteq \mathbb{P}V^\star$$

with torus part $X_W^\circ \coloneqq X_W \cap \mathbb{T}_{E^\star}$.

Recall that a *very affine variety* is a closed irreducible subvariety of an algebraic torus.

Proposition 2.13 ([DSW21, Prop. 3.8])**.** *If $W \subseteq V$ is a connected configuration, then $X_W^\circ \subseteq \mathbb{T}_{E^\star}$ is a very affine variety with closure $\overline{X_W^\circ} = X_W$.*

Theorem 2.14 ([Pat10, Thm. 4.1], [DSW21, Main Thm.])**.** *Let $W \subseteq V$ be a configuration with matroid $\mathsf{M} = \mathsf{M}_W$ Then the non-smooth loci of the configuration hypersurfaces are given by*

$$\hat{X}_W^{\mathrm{nsm}} = \{\beta \in V^\star \mid \operatorname{rank}(\beta \circ Q_W) < \operatorname{rank} \mathsf{M} - 1\}, \quad X_W^{\mathrm{nsm}} = \mathbb{P}X_W^{\mathrm{nsm}}.$$

If M is connected and $\operatorname{rank} \mathsf{M} \geq 2$, then the codimensions in $V^\star$ and $\mathbb{P}V^\star$ equal 3.

3. Bloch's incidence variety

A first step towards constructing an explicit resolution of singularities of X_W appeared in [BEK06, §4] and later in work of Bloch [Blo; Blo20]. We recall the latter in § 3.1 and compare with the former in § 3.7.

3.1. Bloch's definitions.

Bloch associated to a configuration $W \subseteq V$ an incidence subvariety Λ_W of $\mathbb{P}^{r-1} \times \mathbb{P}V^\star$, where $\mathbb{P}^{r-1} \cong \mathbb{P}W$ by a choice of basis.

Definition 3.1. For a configuration $W \subseteq V$, consider the affine variety

$$\hat{\Lambda}_W := \{(w, \beta) \in W \times V^\star \mid Q^\star_W(w, \beta) = 0\} \xhookrightarrow{\ell \times \mathrm{id}} V \times V^\star.$$

The biprojectivization of $\left\{(w, \beta) \in \hat{\Lambda}_W \mid w, \beta \neq 0\right\}$ defines the variety

$$\Lambda_W \subseteq \mathbb{P}W \times X_W \subseteq \mathbb{P}W \times \mathbb{P}V^\star \xhookrightarrow{\mathbb{P}\ell \times \mathrm{id}} \mathbb{P}V \times \mathbb{P}V^\star.$$

We will without notice move freely between $\hat{\Lambda}_W$ and Λ_W on one side, and their images under the inclusions induced by ℓ on the other. ♢

Using the matrices (**2.10**), both $\hat{\Lambda}_W$ and Λ_W are defined explicitly by the r equations

$$\begin{aligned} \hat{\Lambda}_W &= \left\{(w, \beta) \mid AD_{[\beta]}A^\intercal[w] = 0\right\} \\ &= \left\{(w, \beta) \mid AD_{A^\intercal[w]}[\beta] = 0\right\}, \end{aligned} \tag{3.2}$$

where $D_{[v]}$ denotes the $n \times n$ diagonal matrix with entries $(D_v)_{i,i} = v_i$.

Bloch also gave an alternative more abstract description of Λ_W as a scheme. Consider the coherent sheaf $\mathscr{F}$ on $\mathbb{P}W$ defined by the presentation

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}W} \otimes_{\mathbb{K}} W \xrightarrow{\varphi} \mathscr{O}_{\mathbb{P}W}(1) \otimes_{\mathbb{K}} V \longrightarrow \mathscr{F} \longrightarrow 0, \tag{3.3}$$

where

$$\varphi(1 \otimes w) = Q_W(w, -) = \sum_{i \in E} w_i \ell_i \otimes x_i \in W^\star \otimes V$$

The injectivity of φ is due to the fact that the sections $\ell_i \otimes x_i$ of $\mathscr{O}_{\mathbb{P}W}(1) \otimes_{\mathbb{K}} V$ are locally $\mathscr{O}_{\mathbb{P}W}$-linearly independent.

Proposition 3.4 ([Blo20, Prop. 3.5.(i)])**.** *For any configuration $W \subseteq V$, there is an isomorphism of schemes $\Lambda_W \cong \operatorname{Proj} \operatorname{Sym}(\mathscr{F})$.* □

Bloch deduces various properties of Λ_W using the two projections

$$\begin{array}{ccccc} \mathbb{P}V & \xleftarrow{p_1} & \mathbb{P}V \times \mathbb{P}V^\star & \xrightarrow{p_2} & \mathbb{P}V^\star \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid \\ \mathbb{P}W & \xleftarrow{p_1} & \mathbb{P}W \times X_W & \xrightarrow{p_2} & X_W \end{array} \tag{3.5}$$

3.2. **Stratification by flats and irreducibility.** The first projection in (3.5) induces a stratification of Λ_W by flats. For any flat F of M_W, consider the (affine) strata in $\mathbb{P}W$ and Λ_W indexed by F:

$$U_F \coloneqq \mathbb{P}(W/F)^\circ, \quad \Lambda_W|_{U_F} \coloneqq \Lambda_W \cap p_1^{-1}(U_F).$$

Note that $0 \neq w \in W$ has projective image in U_F exactly if $F = F(w)$ (see (2.2)).

Lemma 3.6. *Let $W \subseteq V$ be a configuration. For any flat F of M_W and $|\mathbb{K}| \gg 0$, $U_F \neq \varnothing$ exactly if F is proper. In this case, $\dim U_F = \operatorname{rank} \mathsf{M} - \operatorname{rank}(F) - 1$.*

Proof. Note that $U_E = \varnothing$. If F is proper, then M_W/F contains no loops (see [Oxl11, §3.1, Ex. 8.(a)]). This means that $\ell_i \neq 0$ for all $i \in E\backslash F$ and hence $U_F \neq \varnothing$ for $|\mathbb{K}| \gg 0$. The dimension statement follows from the definition of U_F. □

As a direct consequence of the description of Λ_W in (3.2) we record the following lower bound.

Lemma 3.7. *For any flat F of M_W, the product $U_F \times V(x_i \mid i \in E\backslash F)$ is contained in Λ_W. In particular, $\Lambda_W|_{U_F}$ is nonempty if U_F is nonempty and $F \neq \varnothing$.*

Proof. For any $w \in W$ with projective image in U_F, we have $(A^\intercal[w])_j = 0$ if and only if $j \in F$. For such a w and $\beta \in V(x_i \mid i \in E\backslash F)$ then $D_{A^\intercal[w]}[\beta] = 0$. The claimed containment thus follows from (3.2). □

Bloch [Blo20, §4] shows that $\mathscr{F}$ is a vector bundle on the strata defined by flats.

Proposition 3.8. *For any configuration $W \subseteq V$ and any flat F of $\mathsf{M} = \mathsf{M}_W$, there is an isomorphism of schemes*

$$\Lambda_W|_{U_F} \cong U_F \times \mathbb{P}^{n_F - 1}, \quad n_F \coloneqq |E| - \operatorname{rank}(\mathsf{M}\backslash F).$$

If M is round, then Λ_W is a projective bundle of rank $n_F = |E| - \operatorname{rank} \mathsf{M}$ over $\mathbb{P}W$.

Proof. Choose a basis $B = \{i_1, \ldots, i_m\}$ of $\mathsf{M}\backslash F$ and vectors $w^1, \ldots, w^m \in W$ that project under $\pi_{E\backslash F}$ to a basis of $W\backslash F$ such that $w^j_{i_k} = \delta_{j,k}$. Note that U_F lies in the affine chart of $\mathbb{P}(W/F)$ defined by $\ell_k = 1$, where $k \in E\backslash F$. This identifies

$$\mathscr{O}_{U_F}(1) \cong \mathscr{O}_{U_F}, \quad \ell_i \mapsto \frac{\ell_i}{\ell_k}.$$

Since $\frac{\ell_i}{\ell_k}|_{U_F} = 0$ for $i \in F$, we find that

$$\varphi(1 \otimes w)|_{U_F} = \sum_{i \in E\backslash F} w_i \frac{\ell_i}{\ell_k}|_{U_F} \otimes x_i.$$

In particular, $\varphi(1 \otimes (W \cap \mathbb{K}^F))|_{U_F} = 0$ where $\mathbb{K}^F = \ker \pi_{E\backslash F}$. Restricted to U_F, the image of φ is thus generated of $\mathscr{O}_{U_F}$ by

$$\varphi(1 \otimes w^j)|_{U_F} \equiv \frac{\ell_{i_j}}{\ell_k} \otimes x_{i_j} \mod \langle 1 \otimes x_i \mid i \in E\backslash B \rangle, \quad j = 1, \ldots, m.$$

Since $\frac{\ell_i}{\ell_k} \in \mathbb{K}[U_F]^\times$ is a unit for each $i \in E\backslash F$, this shows that $\mathscr{F}|_{U_F} \cong \mathscr{O}_{U_F}^{n_F}$. Applying $\operatorname{Proj}\operatorname{Sym}$ yields $\Lambda_W|_{U_F} \cong \mathbb{P}^{n_F-1}_{U_F} = U_F \times \mathbb{P}^{n_F-1}$ as claimed.

Suppose now that M is round and hence $n_F = |E| - \operatorname{rank}\mathsf{M}$ for all flats F. Then B and $w^1, \dots, w^m \in W$ are bases of, respectively, M and W, and $\frac{\ell_i}{\ell_k} \in \mathscr{O}^\times_{\mathbb{P}W,w}$ is a unit for each $i \in B$ and $w \in U_F$. As above, we find that $\mathscr{F}_w \cong \mathscr{O}^{n_F}_{\mathbb{P}W,w}$ for all $w \in \mathbb{P}W$. It follows that Λ_W is a projective bundle of rank n_F as claimed. □

Bloch [Blo20, Prop. 3.5.(ii)] shows that Λ_W is an irreducible complete intersection.

Corollary 3.9. *Let $W \subseteq V$ be a configuration with matroid $\mathsf{M} = \mathsf{M}_W$. Then Λ_W is a complete intersection of codimension* $\operatorname{rank}\mathsf{M}$ *in $\mathbb{P}W \times \mathbb{P}V^\star$. If M is connected and $|\mathbb{K}| \gg 0$, then Λ_W is an integral scheme, the closure of the open stratum $\Lambda_W|_{U_\varnothing}$. Conversely, if Λ_W is irreducible and M is loopless, then M must be connected.*

Proof. By Lemma 3.6 and Proposition 3.8 the stratum of Λ_W indexed by any proper flat F of M has dimension

$$\begin{aligned}\dim(\Lambda_W|_{U_F}) &= \dim(W/F) - 1 + |E| - \operatorname{rank}(\mathsf{M}\backslash F) - 1\\ &= |E| + \operatorname{rank}(\mathsf{M}/F) - \operatorname{rank}(\mathsf{M}\backslash F) - 2.\end{aligned}$$

Its codimension in $\mathbb{P}W \times \mathbb{P}V^\star$ therefore equals (see [Oxl11, Prop. 3.1.6])

$$\begin{aligned}\operatorname{codim}_{\mathbb{P}W\times\mathbb{P}V^\star}(\Lambda_W|_{U_F}) &= \operatorname{rank}\mathsf{M} - \operatorname{rank}(\mathsf{M}/F) + \operatorname{rank}(\mathsf{M}\backslash F)\\ &= \operatorname{rank}(F) + \operatorname{rank}(\mathsf{M}\backslash F) \geq \operatorname{rank}\mathsf{M}.\end{aligned} \tag{3.10}$$

In particular, $\operatorname{codim}_{\mathbb{P}W\times\mathbb{P}V^\star}\Lambda_W \geq \operatorname{rank}\mathsf{M}$. Since Λ_W can be defined by $\operatorname{rank}\mathsf{M}$ equations, it follows that it is a complete intersection of codimension $\operatorname{rank}\mathsf{M}$. In particular, it is equidimensional and has no embedded components.

Again by Lemma 3.6 and Proposition 3.8, the open stratum $\Lambda_W|_{U_\varnothing}$ indexed by the empty flat is nonempty, reduced and irreducible. Its closure is thus a reduced irreducible component of Λ_W. If Λ_W is irreducible, it is therefore reduced.

Irreducibility of Λ_W occurs exactly if the other strata do not contribute additional components, which is to say that the inequality (**3.10**) be strict for all nonempty proper flats. This strictness is equivalent to no such flat being a separator of M (see [Oxl11, Prop. 4.2.1]). Thus, Λ_W is irreducible if and only if M only allows the trivial separators $\varnothing$ and M, which is equivalent to M being connected since separators are by definition unions of components.

For the converse, note that the complement of a loop is a separator but not a flat, whereas all separators of a loopless matroid are flats (see [Oxl11, §4.1, Ex. 2]). □

3.3. **Normality and smoothness.** We now describe the Jacobian matrix of Bloch's incidence variety and find smooth points over the strata by flats. These are then used to establish normality using Serre's criterion and to relate smoothness to roundness of the matroid.

Lemma 3.11. *Let $W \subseteq V$ be a configuration with matroid $\mathsf{M} = \mathsf{M}_W$. Then the (transposed) Jacobian matrix $J_W(w,\beta)$ at $(w,\beta) \in W \times V^\star$ obtained from the equations for $\hat{\Lambda}_W$ in* (**3.2**) *reads*

$$J_W(w,\beta) = \left(\ AD_{[\beta]}A^\intercal \ \middle|\ AD_{A^\intercal[w]}\ \right).$$

Its column space equals that of

$$\left(\begin{array}{c|c} \sum_{i\in F}\beta_i[\ell_i][\ell_i]^\intercal & A_{E\setminus F} \end{array} \right) = \left(\begin{array}{c|c} [\beta_F\circ Q_{W|_F}] & A_{E\setminus F} \end{array} \right),$$

where $F = F(w)$ *and the matrix* $A_{E\setminus F}$ *is obtained from* A *by removing the columns indexed by* F*. In particular,*

$$\operatorname{rank}(\mathsf{M}\setminus F) \le \operatorname{rank} J_W(w,\beta) \le \operatorname{rank}(\{i\in F \mid \beta_i\ne 0\}\cup E\setminus F),$$

and when $\beta_F \notin \hat{X}_{W|_F}$ *the matrix* $J_W(w,\beta)$ *has full rank:* $\operatorname{rank} J_W(w,\beta) = \operatorname{rank}\mathsf{M}$.

Proof. The block matrix expression for $J_W(w,\beta)$ is immediate from (**3.2**). Note that

$$AD_{[\beta]}A^\intercal = \sum_{i\in E}\beta_i[\ell_i][\ell_i]^\intercal$$

is a sum of square matrices of size $\operatorname{rank}\mathsf{M}$. The jth column of $\beta_i[\ell_i][\ell_i]^\intercal$ is a scaling of $[\ell_i]$ by $\beta_i[\ell_i]_j$. If $F = F(w)$, then $(A^\intercal[w])_j = \ell_j(w) = 0$ if and only if $j\in F$ (by (**2.2**)). That is, the nonzero columns of $AD_{A^\intercal[w]}$ are indexed by $E\setminus F$ are nonzero and agree with the corresponding ones of A up to $\mathbb{K}^\times$-rescaling. It follows that the column space of the second block of $J_W(w,\beta)$ agrees with that of $A_{E\setminus F}$. Using column operations, the summands with indices in $E\setminus F$ in the first block can then be eliminated. This gives the equality

$$\sum_{i\in F}\beta_i[\ell_i][\ell_i]^\intercal = [\beta_F\circ Q_{W|_F}],$$

and the claimed inequalities follow. For all $\beta_F\notin \hat{X}_{W|_F}$, the latter matrix has (full) rank equal to $\operatorname{rank}\mathsf{M}|_F = \operatorname{rank} A_F$, so $J_W(w,\beta)$ does too. □

Lemma 3.12. *Let* $W\subseteq V$ *be a configuration with matroid* $\mathsf{M} = \mathsf{M}_W$*. Then* Λ_W *has smooth points over* U_F*, that is,* $\Lambda_W^{\mathrm{sm}}|_{U_F}\ne\varnothing$*, for any proper flat* $F\ne\varnothing$ *of* M *if* $|\mathbb{K}|\gg 0$.

Proof. Lemma 3.6 yields a $w\in W$ with projective image in U_F. Since $F\ne\varnothing$ and $|\mathbb{K}|\gg 0$, there is an element $\beta'\notin\hat{X}_{W|_F}$ in the complement of $\hat{X}_{W|_F}$. Define $\beta\in\mathbb{P}V^\star$ by $\beta_F := \beta'$ and $\beta_{E\setminus F} := 0$. Due to Lemmas 3.7 and 3.11, then $(w,\beta)\in\hat{\Lambda}_W$ and $J_W(w,\beta)$ has full rank. It follows that (w,β) is a smooth point of $\hat{\Lambda}_W$. Its projective image in $\mathbb{P}W\times\mathbb{P}V^\star$ is then a smooth point of Λ_W over U_F. □

Corollary 3.13. *Let* $W\subseteq V$ *be a configuration with connected matroid* $\mathsf{M} = \mathsf{M}_W$*. Then* Λ_W *is an integral normal scheme.*

Proof. By hypotheses on M and Corollary 3.9, Λ_W is a reduced complete intersection integral scheme. By using Serre's criteria for normality and reducedness (applied in affine charts), it is enough to check that Λ_W is regular in codimension one. By way of contradiction, suppose that Λ_W has an irreducible subvariety Z of codimension one with generic point in Λ_W^{nsm}. By Proposition 3.8, $\Lambda_W|_{U_\varnothing}$ is smooth and hence $Z\subseteq\overline{\Lambda_W|_{U_F}}$ for some proper flat $F\ne\varnothing$, where $\Lambda_W|_{U_F}$ is irreducible. The strict inequality (**3.10**) in the proof of Corollary 3.9 shows that $\Lambda_W|_{U_F}$ has codimension at least one. It follows that $Z = \overline{\Lambda_W|_{U_F}}$ and then

$$\Lambda_W^{\mathrm{sm}} \supseteq \Lambda_W^{\mathrm{sm}}|_{U_F} \subseteq \Lambda_W|_{U_F} \subseteq \overline{\Lambda_W|_{U_F}} = Z \subseteq \Lambda_W^{\mathrm{nsm}} = \Lambda\setminus\Lambda_W^{\mathrm{sm}}$$

implies $\Lambda_W^{\mathrm{sm}}|_{U_F} = \varnothing$, which contradicts Lemma 3.12 if $|\mathbb{K}| \gg 0$.

Finally, by base change, we may assume that $\mathbb{K}$ is infinite, since normality descends along faithfully flat ring maps (see [Sta23, Tag 033G]). □

Proposition 3.8 shows in particular that Λ_W is smooth if M_W is round. We now establish a refined converse statement: the non-smooth strata arise exactly from flats that violate roundness.

Theorem 3.14. *Let $W \subseteq V$ be a configuration with connected matroid $\mathsf{M} = \mathsf{M}_W$. For any proper flat F of M and $|\mathbb{K}| \gg 0$, $\Lambda_W^{\mathrm{nsm}}|_{U_F} \neq \varnothing$ if and only if $\operatorname{rank}(\mathsf{M}\backslash F) < \operatorname{rank}\mathsf{M}$. In particular, Λ_W is smooth if and only if M is round.*

Proof. Lemma 3.6 yields a $w \in W$ with projective image in U_F.

First, suppose that $\operatorname{rank}(\mathsf{M}\backslash F) < \operatorname{rank}\mathsf{M}$. Since M is connected, $E\backslash F$ is not a flat and hence $\operatorname{rank}(E\backslash F) = \operatorname{rank}(\{j\} \cup (E\backslash F))$ for some $j \in F$. Define $\beta \in V^\star$ by $\beta_i = \delta_{i,j}$. By Lemmas 3.7 and 3.11, $(w, \beta) \in \hat{\Lambda}_W|_{U_F}$ with

$$\operatorname{rank} J_W(w,\beta) \leq \operatorname{rank}(\{j\} \cup E\backslash F) = \operatorname{rank}(\mathsf{M}\backslash F) < \operatorname{rank}\mathsf{M}.$$

The projective image of (w, β) thus lies in Λ_W^{nsm}.

Now, suppose that $\operatorname{rank}(\mathsf{M}\backslash F) = \operatorname{rank}\mathsf{M}$. Represent an arbitrary point of $\Lambda_W|_{U_F}$ as the projective image of $(w, \beta) \in \hat{\Lambda}_W$. By Lemma 3.11, $\operatorname{rank} J_W(w,\beta) \geq \operatorname{rank}(E\backslash F) = \operatorname{rank}\mathsf{M}$. The projective image of (w, β) thus lies in Λ_W^{sm}. □

3.4. **An example.** The following example illustrates the statement of Theorem 3.14.

Example 3.15. If $\mathsf{M} = \mathsf{M}_G$ is the matroid of a graph G and $W = W_G$ the configuration resulting from the (pruned) incidence matrix to an arbitrary orientation (see [DSW21, §2.4]), we replace the index W by G. For the graph G in Figure 1, consider the connected graphic matroid $\mathsf{M} = \mathsf{M}_G$ with $n = 5$ and $r = 3$, and the flats $F_1 = \{1, 2, 4\}$ and $F_2 = \{1, 3, 5\}$. Then W_G is the row span of the matrix

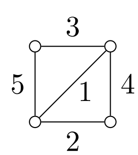


FIGURE 1. A graph with non-round matroid.

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

For $i \in \{1, 2\}$, we have $\operatorname{rank}(\mathsf{M}\backslash F_i) < 3$ and hence M is not round. One computes

$$[Q_G] = \begin{pmatrix} x_1 + x_4 + x_5 & x_4 & x_5 \\ x_4 & x_2 + x_4 & 0 \\ x_5 & 0 & x_3 + x_5 \end{pmatrix} \quad \text{and}$$

$$\Psi_G = x_1x_2x_3 + x_1x_3x_4 + x_2x_3x_4 + x_1x_2x_5 + x_2x_3x_5 + x_1x_4x_5 + x_2x_4x_5 + x_3x_4x_5.$$

Using [DSW21, Lem. 4.22], we find a decomposition of the non-smooth locus of X_G into irreducible components

$$X_G^{\mathrm{nsm}} = C_{F_1} \cup C_{F_2} \quad \text{where} \quad C_F \coloneqq \mathrm{V}(\Psi_{G_F}, \{x_i \mid i \notin F\})$$

and G_F denotes the induced subgraph with edges F. For both handles $\{2,4\}$ and $\{3,5\}$, any irreducible component C different from both C_{F_i} for $i \in \{1,2\}$ must be of type (c) in loc. cit. But this entails $C \subseteq V(x_2, x_3, x_4, x_5)$ and contradicts the codimension statement in Theorem 2.14. By symmetry then both components C_{F_i} for $i \in \{1,2\}$ must occur.

Clearly, $C_{F_1} \cap C_{F_2} = \{\beta_0\}$ where $\beta_0 \coloneqq (1:0:0:0:0)$. For $\beta \in X_G^{\mathrm{nsm}}$, the matrix $\beta[Q_G]$ has rank 1, so $p_2^{-1}(\beta) \cong \mathbb{P}^1$ (recall diagram (**3.5**)). In particular, $p_1 \circ p_2^{-1}(\beta_0)$ is the line through

$$\alpha_{F_1} \coloneqq (0:0:1:0:1) \quad \text{and} \quad \alpha_{F_2} \coloneqq (0:1:0:1:0),$$

and $(\alpha_{F_i}, \beta) \in p_2^{-1}(\beta)$ for each $i \in \{1,2\}$ and $\beta \in C_{F_i}$: that is, $p_1 \circ p_2^{-1}(C_{F_i})$ is a pencil of lines through α_{F_i}. We find that Λ_G^{nsm} consists of the two points (α_{F_i}, β_0) for $i \in \{1,2\}$.

It is exactly for these flats that the "round" condition fails: $Q_G^\star(w, -)$ has rank 2 for $w = \alpha_{F_i}$ where $i \in \{1,2\}$, and rank 3 otherwise. Taking the kernel for each w, we compute for each $i \in \{1,2\}$ that $p_1^{-1}(\alpha_{F_i}) = \mathbb{P}\mathbb{K}^{F_i} \cong \mathbb{P}^2$ is the coordinate subspace containing the curve C_{F_i}, and $p_1^{-1}(w) \cong \mathbb{P}^1$ otherwise. We will revisit this example in §3.7 and §4.5. ♢

3.5. **A candidate resolution.** We now focus on the second projection in (**3.5**). Bloch [Blo20, Prop. 3.5.(ii)] showed that $p_2 \colon \Lambda_W \to X_W$ is birational. More precisely:

Proposition 3.16. *Let $W \subseteq V$ be a connected configuration. Then the second projection $p_2 \colon \Lambda_W \to X_W$ is an isomorphism over the smooth locus X_W^{sm}. In particular, it is proper birational, and hence surjective.*

Proof. By Theorem 2.14, we have $\beta \in X_W^{\mathrm{sm}}$ if and only if $\beta \circ Q_W$ has corank 1. We may thus assume that for some basis $w^1, \dots, w^r$ of W the matrix $(Q_W(w^i, w^j))_{1 \le i,j < r}$ has full rank in an open neighborhood U of β in X_W. Over U, Λ_W is then given in terms of homogeneous coordinates $z = z_1, \dots, z_r$ of $\mathbb{P}W$ by linear equations:

$$\Lambda_W \cap p_2^{-1}(U) = V\Big(\sum_{j<r} Q_W(w^i, w^j) z_i + Q_W(w^r, w^j) z_r \mid j = 1, \dots, r-1\Big) \tag{3.17}$$

In particular, it maps to the chart $\{z_r = 1\}$ of $\mathbb{P}W$. The unique solution

$$-(Q_W(w^i, w^j)_{1 \le i,j < r})^{-1} (Q_W(w^r, w^j)_{1 \le j < r},$$

of the equations in (**3.17**) for $z_r = 1$ defines a morphism $\varphi \colon U \to \{z_r = 1\}$ such that $(\varphi, \mathrm{id}_U) \colon U \to \{z_r = 1\} \times U$ is an inverse of p_2 over U. □

We obtain the following consequence of Propositions 3.8 and 3.16.

Corollary 3.18. *For any configuration $W \subseteq V$ whose matroid is round, the second projection $p_2 \colon \Lambda_W \to X_W$ is a resolution of singularities that only modifies the singular points of X_W.*

3.6. **Singularities of the affine cone.** Suppose a basis for W has been chosen as in (2.1), and introduce coordinate functions $u = u_1, \dots, u_r$ relative to that basis. The projective incidence variety Λ_W is cut out by the ideal

$$I_W \coloneqq (q_1, \dots, q_r) \subseteq \operatorname{Sym}(W^\star) \otimes \operatorname{Sym}(V) = \mathbb{K}[u, x]$$

generated by the r quadrics

$$q_i \coloneqq (A[Q_E]A^\intercal[u])_i \text{ with } 1 \leq i \leq r \text{ and } [u] = (u_1, \dots, u_r)^\intercal,$$

homogeneous separately in u and in $x = x_1, \dots, x_n$. By row reduction, and suitably renumerating E, there is an $r \times n$ matrix $A = (\mathrm{id}_r \,|B)$ whose row span is also W, and this replacement of one matrix with row span W by another corresponds to a coordinate change purely in u. Observe that now the (i, i)-entry of $A[Q_E]A^\intercal$ is x_i plus a $\mathbb{K}$-linear combination of $x_{r+1}, \dots, x_n$, while no off-diagonal entry contains any of $x_1, \dots, x_r$. In particular,

$$q_i = x_i u_i + \sum_{j=r+1}^{n} \tilde{u}_{i,j} x_j$$

where $\tilde{u}_{i,j}$ is a $\mathbb{K}$-linear combination of $u_1, \dots, u_r$.

Introduce a term order on $\mathbb{K}[u, x]$ that refines the lex-order on x. Then the lead term of q_i is $x_i u_i$. In particular, the lead terms of the q_i are square-free and relatively prime.

The relative primeness implies that these elements form a Gröbner basis under the chosen term order (because any s-pair made from polynomials with relatively prime lead terms reduces to zero via just those two polynomials; immediate termination of the Buchberger algorithm shows completion of the Gröbner basis search). This in turn implies that the initial ideal of I_W is $(x_1 u_1, \dots, x_r u_r)$. Since the initial ideal is a complete intersection of height r, so is I_W (on the generators $\{q_i\}$), and in particular I_W is equidimensional. Moreover, since $(x_1 u_1, \dots, x_r u_r)$ is a radical ideal, so is I_W.

3.6.1. *Linkage.* The height r prime $I_{W,u} \coloneqq (u_1, \dots, u_r)$ is an associated prime of the (equidimensional) complete intersection I_W of height r. Let

$$I_{W,0} \coloneqq I_W : I_{W,u}$$

be the ideal quotient and denote by $\mathrm{V}(-)$ the affine schemes attached to ideals in a ring. Since I_W is radical, $I_{W,0}$ is the radical ideal to the union of the components of $\mathrm{V}(I_W)$ different from $\mathrm{V}(I_{W,u})$.

Let $I_{W,x} = (x_1, \dots, x_n)$; it follows from Corollary 3.9, that $\mathrm{V}(I_W) \setminus (\mathrm{V}(I_{W,u}) \cup \mathrm{V}(I_{W,x}))$ is integral over (the infinite field) $\bar{\mathbb{K}}$ and hence over $\mathbb{K}$ as well. Equidimensionality of I_W implies that no associated prime of I_W can strictly contain $I_{W,u}$. If $n > r$, no associated prime can be equal to, much less contain, $I_{W,x}$. It follows that $I_{W,0}$ is in fact a prime ideal when the matroid is connected.

Definition 3.19. Let $\hat{\Lambda}_{W,0} \subseteq W \times V^\star \overset{\ell}{\hookrightarrow} V \times V^\star$ be the affine variety defined by $I_{W,0}$. Note that the associated biprojective variety is exactly Λ_W. ♢

By construction, $I_{W,u} = I_W : I_{W,0}$ and $I_{W,0} = I_W : I_{W,u}$. Two pure-codimensional ideals $\mathfrak{a}, \mathfrak{b}$ of equal codimension in a ring R are said to be in *direct linkage* if there is a third R-ideal $\mathfrak{c}$ of the same codimension with: $\mathfrak{c}$ is a complete intersection; $\mathfrak{c} : \mathfrak{a} = \mathfrak{b}, \mathfrak{c} : \mathfrak{b} = \mathfrak{a}$. Thus, $I_{W,u}$ and $I_{W,0}$ are (directly) linked via the complete intersection I_W. *Linkage* is the equivalence relation between ideals generated by direct linkage. The notions go back to Peskine and Szpiro [PS74] and force strong relations between the three ideals. For example, $R/\mathfrak{a}$ is Cohen–Macaulay if and only if $R/\mathfrak{b}$ is so (see [PS74, Prop. 1.3]).

For directly linked ideals, $\mathrm{Hom}_{R/\mathfrak{c}}(R/\mathfrak{a}, R/\mathfrak{c})$ identifies in the obvious way with $(\mathfrak{c} : \mathfrak{a})/\mathfrak{c} = \mathfrak{b}/\mathfrak{c}$. Suppose that $R/\mathfrak{a}, R/\mathfrak{b}$ are Cohen–Macaulay rings. As $\mathfrak{c}$ is a complete intersection, $R/\mathfrak{c}$ is isomorphic to the canonical module of $R/\mathfrak{c}$ and hence $\mathfrak{b}/\mathfrak{c} = \mathrm{Hom}_{R/\mathfrak{c}}(R/\mathfrak{a}, \omega_{R/\mathfrak{c}})$ is a canonical module for $R/\mathfrak{a}$.

In the case at hand, $\mathfrak{c} = I_W$, $\mathfrak{a} = I_{W,u}$, $\mathfrak{b} = I_{W,0}$. Since $I_{W,u}$ is a complete intersection in its own right, $I_{W,0}$ is *licci* (in the *li*nkage *c*lass of a *c*omplete *i*ntersection; see [PS74; HU87]). In particular, $I_{W,0}/I_W$ is isomorphic to the canonical module of $\mathbb{K}[u,x]/I_{W,u}$, which is (up to shift) $\mathbb{K}[u,x]/I_{W,u}$ itself. It follows that $I_{W,0} = I_W + (f)$ for a suitable $f \in \mathbb{K}[u,x]$.

Writing $[q] = (q_1, \ldots, q_r)$ and $R = \mathbb{K}[u,x]$, we now consider the free resolutions (Koszul complexes, both of length r) to I_W and $I_{W,u}$, and the morphism induced between them by the equation $[q] = A[Q_E]A^\intercal[u]$:

$$\begin{array}{ccccccccc}
\bigwedge^r R^r & \xrightarrow{\wedge^r([q])} & \cdots & \xrightarrow{\wedge^3([q])} & \bigwedge^2 R^r & \xrightarrow{\wedge^2([q])} & \bigwedge^1 R^r & \xrightarrow{[q]} & \bigwedge^0 R^r = R \\
\downarrow{\scriptstyle \wedge^r(A[Q_E]A^\intercal)} & & & & \downarrow{\scriptstyle \wedge^2(A[Q_E]A^\intercal)} & & \downarrow{\scriptstyle A[Q_E]A^\intercal} & & \downarrow{\scriptstyle 1} \\
\bigwedge^r R^r & \xrightarrow{\wedge^r([u])} & \cdots & \xrightarrow{\wedge^3([u])} & \bigwedge^2 R^r & \xrightarrow{\wedge^2([u])} & \bigwedge^1 R^r & \xrightarrow{[u]} & \bigwedge^0 R^r = R
\end{array}$$

Since the rows of this diagram resolve R/I_W and $R/I_{W,u}$ respectively, the total complex is the mapping cone to the projection $R/I_W \to R/I_{W,u}$ and hence naturally identifies with $I_{W,u} : I_W$. Since $R/I_{W,u}$ is Cohen–Macaulay, so is $R/I_{W,0}$ and $I_{W,u} : I_W$ is its canonical module. Thus, the dual of the diagram above yields a free resolution of $R/I_{W,0}$. It is not minimal, but by pruning the vertical isomorphism in the rightmost column we obtain a bigraded minimal resolution as the total complex of the following double complex (with $\deg(u_i) = (0,1)$ and $\deg(x_j) = (1,0)$):

(3.20)

$$\begin{array}{ccccccc}
(\bigwedge^r R^r)(0,0) & \xleftarrow[\wedge^r([q])]{} & \cdots & \xleftarrow[\wedge^3([q])]{} & (\bigwedge^2 R^r)(2-r,2-r) & \xleftarrow[\wedge^2([q])]{} & (\bigwedge^1 R^r)(1-r,1-r) \\
\uparrow{\scriptstyle \wedge^r(A[Q_E]A^\intercal)} & & & & \uparrow{\scriptstyle \wedge^2(A[Q_E]A^\intercal)} & & \uparrow{\scriptstyle A[Q_E]A^\intercal} \\
(\bigwedge^r R^r)(-r,0) & \xleftarrow[\wedge^r([u])]{} & \cdots & \xleftarrow[\wedge^3([u])]{} & (\bigwedge^2 R^r)(-r,2-r) & \xleftarrow[\wedge^2([u])]{} & (\bigwedge^1 R^r)(-r,1-r)
\end{array}$$

Thus, the free module F_i of the $\mathbb{Z}$-graded resolution $F_\bullet \twoheadrightarrow R/I_{W,0}$, with $1 \le i \le r-1$, is given by $F_i = (R(-2i))^{\binom{r}{i}} \oplus (R(-r-i+1))^{\binom{r}{i-1}}$, while $F_r = (R(1-2r))^{\binom{r}{r-1}}$. Since the largest shift in any F_i is $2r-1$, and since the canonical module of R is generated in degree $-(r+n)$, the a-invariant of $R/I_{W,0}$ is $(2r-1)-(r+n) = r-1-n < 0$.

We have proved most of the following result.

Theorem 3.21. *The coordinate ring $\mathbb{K}[u,x]/I_{W,0}$ of $\hat{\Lambda}_{W,0}$ is a normal Cohen–Macaulay domain of type r and of projective dimension r over $\mathbb{K}[u,x]$. It has Castelnuovo–Mumford regularity $r-1$ and the defining ideal is $I_{W,0} = I_W + \langle \psi_W \rangle$.*

Proof. The last statement follows from $\bigwedge^r(A[Q_E]A^\intercal) = [\psi_W]$ and is in fact a special case of [Wie69, Satz 2].

Normality of $\hat{\Lambda}_{W,0}$ follows along the same lines as that of Λ_W. Indeed, the proof of Corollary 3.13 implies that $\hat{\Lambda}_{W,0}$ is normal outside the vanishing locus of u. On the other hand, the Jacobian of the height r ideal $(A[Q_E]A^\intercal[u], \psi_W)$ in a point where u vanishes is

$$\begin{pmatrix} 0 & \nabla\psi_W \\ A[Q_E]A^\intercal & 0 \end{pmatrix}.$$

At a point of $\hat{\Lambda}_{W,0}$ where the first coordinate gives a generic point on $\hat{X}_W$, the lower left block has rank $r-1$ and the upper right is nonzero. The claims about regularity and projective dimension are obtained from the minimal resolution (**3.20**). ☐

Remark 3.22. For disconnected M_W, the factorization of ψ_W induces a decomposition of Λ_W and $\hat{\Lambda}_{W,0}$. When $r > 1$, $\hat{\Lambda}_{W,0}$ is not Gorenstein, since its coordinate ring has type r. ♢

F-purity. Let R be a commutative ring of finite characteristic $p > 0$. The Frobenius endomorphism and its iterates

$$\begin{aligned} F^e\colon R &\longrightarrow F^e_*(R) \\ x &\longmapsto x^{p^e} \end{aligned}$$

can be used to classify singularities, where $F^e_*(R)$ is R as an Abelian group, equipped with an R-module structure $r(r') = r^{p^e} r'$. (One traditionally uses $F^e_*(R)$ to indicate the target of the e-th Frobenius on R in order to easily differentiate it from other instances of copies of R).

For an ideal J of R, denote by $J^{[p^e]} \subseteq J^{(p^e)}$ its e-th Frobenius power, the ideal generated by the p^e-th powers of all elements of J.

Definition 3.23. The ring R is *F-pure*, if the Frobenius morphism $F^e\colon R \to F^e_*(R)$ is pure, which is to say that its tensor product with every R-module yields an injective morphism. [2] ♢

The motivation for this definition comes from a theorem of Kunz that states that a ring R of prime characteristic is regular precisely when F^e is flat for some (equivalently all) $e > 0$. In fact, for a local ring, flatness implies that $F^e_*(R)$ is a free module over the image ring $F^e(R)$. In that case, $F(R)$ is even a summand of $F^1_*(R)$ and thus F-pure rings can be viewed as "somewhat similar" to regular rings.

A related condition is *F-injectivity*, determined by the injectivity of the natural Frobenius action on all the local cohomology modules $H^i_\mathfrak{m}(R_\mathfrak{m})$ where $\mathfrak{m}$ runs through the maximal ideals of R. (For this action, view local cohomology as defined via a

[2] The choice of e is mostly immaterial: if F^e is pure for one $e > 0$ then it is so for all $e > 0$. Note in particular that F-purity requires that F is injective itself, hence the ring must be reduced.

Čech complex and take powers of fractions representing classes). In general, F-purity implies F-injectivity and the converse holds for Gorenstein rings.

Remark 3.24. Suppose R is a finitely generated algebra over a field $\mathbb{K}$ of characteristic zero. Lifting R to a finitely generated $\mathbb{Z}$-algebra $\tilde{R}$, one can discuss the properties of its reduction to various primes p.

If $\mathbb{K} = \mathbb{C}$ and X is a reduced finite type scheme over $\mathbb{K}$, a somewhat non-trivial condition (see [Sch09, Definitions 5.1, 6.9]) determines whether X is *du Bois*. While the definition is complicated, it has some appealing consequences. For example, on a proper reduced du Bois scheme X the natural map $H^i(X, \mathbb{C}) \to H^i(X, \mathscr{O}_X)$ is surjective. This is a geometric condition, generalizing the classical Hodge-theoretic facts that for smooth proper complex varieties, and for compact Kähler manifolds, this map is surjective. A very nice introduction to the relationship between du Bois and rational singularities is in Chapter 12 of [Kol95].

It has been shown by Schwede in the same paper [Sch09], that $\mathrm{Spec}(R)$ has du Bois singularities precisely when infinitely many mod-p-reductions of $\tilde{R}$ are F-injective. $\diamondsuit$

An obvious complication for testing whether a given ring is F-pure or not is that one ostensibly needs to check injectivity of infinitely many maps. The following result reduces the problem to a finite computation.

Lemma 3.25 (Fedder's Criterion, [Fed83])**.** *If $(R, \mathfrak{m})$ is a regular local ring of characteristic $p > 0$ and $I \subseteq R$ an ideal then R/I is F-pure precisely when $I^{[p]} : I$ is not contained in $\mathfrak{m}^{[p]}$.* □

There is a version of Fedder's Criterion (of the expected form) for homogeneous ideals in standard graded polynomial rings, [MP22a, Rmk. 2.9].

In general, the ideal quotient $I^{[p]} : I$ is fairly mysterious, even when I is nice. The obvious exception arises when $I = (f_1, \dots, f_k)$ is a complete intersection in a UFD, in which case $I^{[p]} : I$ is generated by $\prod f_i^{p-1}$.

Theorem 3.26. *Both $\hat{\Lambda}_W$ and the affine cone $\hat{\Lambda}_{W,0} = \mathrm{Spec}(\mathbb{K}[u, x]/(I_W + \langle \psi_W \rangle)$ over Λ_W are F-pure over every field of positive characteristic, and du Bois over every field of characteristic zero.*

Proof. Recall that we may preprocess in such a way that the front r columns of A form an identity matrix, and so the i-th component of $A[Q_E]A^\intercal[u]$ has lead term $x_i u_i$ under the monomial order that refines the lex order in x by lex in u.

Let Q be the product of all q_i, and write $\mathfrak{m} \coloneqq (x_1, \dots, x_n, u_1, \dots, u_r)$. Clearly, $Q^{p-1} \in (I_W^{[p]} : I_W)$ and $Q^{p-1} \not\in \mathfrak{m}^{[p]}$, since its lead monomial $\prod_{i=1}^r x_i^{p-1} u_i^{p-1}$ is not in the monomial ideal $\mathfrak{m}^{[p]}$. By Fedder's Criterion, $\mathbb{K}[u, x]/I_W$ is F-pure.

Now, $I_{W,0} = I_W + (\psi_W)$ arises as $I_W : I_{W,u}$, and that is enough to inherit F-purity from $\mathbb{K}[u, x]/I_W$ to $\mathbb{K}[u, x]/I_{W,0}$. Indeed, by [PT24, Lem. 3.1], one infers that $(I_W^{[p]} : I_W)$ is contained in $(I_{W,0}^{[p]} : I_{W,0})$ and thus Q^{p-1} witnesses F-purity of $\mathbb{K}[u, x]/I_{W,0}$ as well. □

3.6.2. *F-rationality and F-regularity.*

Definition 3.27. A ring is *F-finite* if it is a finitely generated module over its Frobenius image. Fields are F-finite precisely when the extension degree $[\mathbb{K}^{1/p}:\mathbb{K}]$ is finite. Finitely generated $\mathbb{K}$-algebras over F-finite fields, as well as their localizations are F-finite, and so are complete local rings with with F-finite residue field.

The F-finite ring R is *strongly F-regular* if for every $c \in R$ not in a minimal prime of R, there exists a natural number $e > 0$ such that the morphism $R \to F^e_*(R)$ that is the composition of the Frobenius followed by multiplication by $F^e_*(c) \in F^e_*(R)$, splits as map of R-modules. ♢

Strongly F-regular rings are Cohen–Macaulay domains, and the property is a local property. A related property is the following.

Definition 3.28. Let R be a local ring R of dimension d with maximal ideal $\mathfrak{m}$, and consider the two conditions

(1) for any c in R not in a minimal prime of R the map from $H^i_{\mathfrak{m}}(R)$ to itself given by multiplication by c following the e-fold iteration of the Frobenius, is injective, for any i;

(2) R is Cohen–Macaulay.

Then R is *F-injective*, if (1) holds, and F-rational if (1) and (2) hold. ♢

All four F-properties localize (see [MP22a]). One notes that F-rationality relates to F-injectivity in a similar way strong F-regularity relates to F-purity. See the diagram at the start of Section 8 in [MP22a] for relative strength of all these properties; strong F-regularity implies all others, and F-purity implies F-injectivity.

Results of K. Smith, N. Hara and K.-I. Watanabe [Smi97; Har98a; HW02] show intricate relations between strong F-regularity and F-rationality on one side and rational and log-terminal singularities on the other.

Below we identify $\hat{\Lambda}_{W,0}$ with its image under ℓ, along the lines of Definition 3.1.

Lemma 3.29. *For algebraically closed $\mathbb{K}$, let $W \subseteq V$, $W^\perp \subseteq V^\star$ be dual configurations on $|E| = n$ elements, of respective ranks r and $n-r$. Then $\hat{\Lambda}_{W,0} \cap (V \times \hat{\mathbb{T}}_{E^\star}) \subseteq V \times V^\star$ and $\hat{\Lambda}_{W^\perp,0} \cap (V^\star \times \hat{\mathbb{T}}_E) \subseteq V^\star \times V$ are isomorphic.*

Proof. Since the varieties are integral, it suffices to describe the isomorphism on closed points.

Let $A, A^\perp$ be full-rank matrices with row spans $W, W^\perp$ respectively, and let $D_{[\beta]}$ be a diagonal matrix as before. Then A and $A^\perp$ minimally span each other's kernel, and $(v, \beta) \in \hat{\Lambda}_{W,0}$ if and only if a) $AD_{[\beta]}A^\intercal$ is rank-deficient, b) v is in the row span of A, and c) $AD_{[\beta]}[v] = 0$.

Assume that $\beta_i \neq 0$ for all $i \in E$ and define a morphism

$$\begin{aligned} \mathbb{D}\colon \hat{\Lambda}_{W,0} \cap (V \times \hat{\mathbb{T}}_{E^\star}) &\longrightarrow \hat{\Lambda}_{W^\perp,0} \cap (V^\star \times \hat{\mathbb{T}}_E), \\ (v, \beta) &\longmapsto (Q_E(\beta, v), 1/\beta), \end{aligned}$$

where $1/\beta = (1/\beta_1, \dots, 1/\beta_n)$, and $Q_E(-,-)$ is the Hadamard product from (**2.8**). Now, a) $A^\perp D_{[1/\beta]}(A^\perp)^\intercal$ is rank-deficient if and only if $AD_{[\beta]}A^\intercal$ is since that is where the determinants $\psi_W(\beta) = \psi_{W^\perp}(1/\beta) \cdot \prod_{i\in E} \beta_i$ vanish. Moreover, if $AD_{[\beta]}[v] = 0$ then

$A[Q_E(\beta, v)] = AD_{[\beta]}[v] = 0$ shows that b) $Q_E(\beta, v)$ is in the row span of $A^\perp$. Finally, v being in the row span of A forces c) $A^\perp D_{[1/\beta]}[Q_E(\beta, v)] = A^\perp[v] = 0$. Hence the image of $\mathbb{D}$ is indeed within $\hat{\Lambda}_{W^\perp,0} \cap (V^\star \times \hat{\mathbb{T}}_E)$. The inverse morphism is given by $(Q_E(\beta^\perp, v^\perp), {\beta^\perp}^{-1}) \leftarrow\!\shortmid (v^\perp, \beta^\perp)$. ☐

The following result of Schwede and Singh will be useful.

Lemma 3.30 ([HMS14, Cor. A4])**.** *An F-finite local ring R is F-rational provided that there exists a regular element $f \in R$ such that $R/\langle f\rangle$ is F-injective and $R[1/f]$ is F-rational.* ☐

Theorem 3.31. *Let W be a configuration for the matroid M over an F-finite field of positive characteristic. If M is connected, then $\hat{\Lambda}_{W,0}$ is F-rational, and in particular normal.*

Proof. It is enough to show this over algebraically closed fields by a result of Vélez [Vél95, p. 440] (see also [DM24]). The normality claim follows from F-rationality in general by [HH94, Thm. 4.2].

By Theorem 3.26, $\hat{\Lambda}_{W,0}$ is F-pure for all W, connected or otherwise, and so every local ring of $\hat{\Lambda}_{W,0}$ is also F-injective for all W. The blueprint of the proof of Theorem 3.31 is the same as the one for [BW24, Thm. 1.2], by induction on the number $n = |E|$. The main tool is Tutte's theorem (see [Oxl11, Thm. 4.3.1]) assuring that for a connected matroid M and any edge $i \in E$, either $\mathsf{M}\backslash i$ or M/i is connected.

The inductive base case is when the rank of M_W is 1. Then it is immediate that the defining ideal of $\hat{\Lambda}_{W,0}$ is generated by ψ_W. But as M is then $\mathsf{U}_{1,n}$, ψ_W is (linear, hence) smooth.

Now suppose that $M\backslash i$ is connected for some $i \in E$. Then i is not a coloop of M and hence $\operatorname{rank}(\mathsf{M}) = \operatorname{rank}(\mathsf{M}\backslash i)$. Thus, $[Q_W]$ and $[Q_{W\backslash i}]$ agree modulo x_i, and likewise the ideal $I_{W,0}$ describing $\hat{\Lambda}_{W,0}$ inside $V \times V^\star$ agrees modulo x_i with $I_{W\backslash i,0}$. In other words, $\hat{\Lambda}_{W\backslash i,0}$ is the hyperplane section $\hat{\Lambda}_{W,0} \cap \mathrm{V}(x_i)$ inside $V \times V^\star$. In the local ring at the origin, F-rationality of $\hat{\Lambda}_{W,0}$ then follows from that of $\hat{\Lambda}_{W\backslash i,0}$ at the origin by [HH94, Thm. 4.2]. This implies that $\hat{\Lambda}_{W,0}$ is F-rational in a neighborhood of the origin since the F-rational locus is open in our rings by a theorem of Vélez, [Vél95]. But since $\hat{\Lambda}_{W,0}$ is defined by a standard graded ideal, F-rationality near the origin implies global F-rationality.

We have reduced to showing that $\hat{\Lambda}_{W,0}$ is F-rational if M and M/i are connected. Since matroid duality exchanges restriction and deletion and also preserves connectedness, $\mathsf{M}^\perp$ has the connected deletion $\mathsf{M}^\perp\backslash i = (\mathsf{M}/i)^\perp$. We may hence assume by induction on $|E|$ that the corresponding $\hat{\Lambda}_{W^\perp,0}$ is F-rational. By Lemma 3.29, the local rings of $\hat{\Lambda}_{W,0}$ in which $\prod_{i\in E} x_i$ is a unit are thus all F-rational.

Let $\hat{\Lambda}^{\mathrm{Frat}}_{W,0}$ be the F-rational locus of $\hat{\Lambda}_{W,0}$. Let $R = \mathbb{K}[V \times V^\star]$, and for a prime ideal $\mathfrak{p}$ containing the defining ideal $I_{W,0}$ of $\hat{\Lambda}_{W,0}$, set

$$\nu(\mathfrak{p}) := \#\{i \in E \mid x_i \in \mathfrak{p}\}.$$

Then define, for $t \in \{0, \dots, n\}$, the sets

$$P_t \coloneqq \{\mathfrak{p} \in \operatorname{Spec}(R/I_{W,0}) \mid \nu(\mathfrak{p}) \leq t\}.$$

The previous paragraph shows for $t = 0$ the implication $[\mathfrak{p} \in P_t] \Rightarrow [\mathfrak{p} \in \hat{\Lambda}^{\mathrm{Frat}}_{W,0}]$. We show now by induction corresponding statements for all t. For, suppose that $\mathfrak{p} \in P_t \setminus P_{t-1}$ and choose $i \in E$ with $x_i \in \mathfrak{p}$. Then the prime ideals $\mathfrak{q}$ in R that correspond to prime ideals in $(R/I_{W,0})[1/x_i]$ have $\nu(\mathfrak{q}) < t$, and it follows from induction that $(R/I_{W,0})_{\mathfrak{q}}$ is F-rational. Since F-rationality is a local property, the ring $(R/I_{W,0})_{\mathfrak{p}}[1/x_i]$ is F-rational. On the other hand, reduction of $(R/I_{W,0})_{\mathfrak{p}}$ by x_i is the localization at $\mathfrak{p}$ of $R/(I_{W,0} + Rx_i)$. But the latter is the local ring of $\hat{\Lambda}_{W\setminus i,0}$ at $\mathfrak{p}(R/(x_i))$, which is F-pure and hence F-injective. By Lemma 3.30, $\hat{\Lambda}_{W,0}$ is F-rational at $\mathfrak{p}$. This concludes the inductive step for the primes in P_t, and hence the proof of the theorem. □

Remark 3.32. Suppose Y is a variety of finite type over a field of characteristic zero, and $\pi\colon \tilde{Y} \to Y$ is a resolution of singularities. Then Y has *rational singularities* if the canonical map to the derived direct image $\mathscr{O}_Y \to R\pi_*(\mathscr{O}_{\tilde{Y}})$ is a quasi-isomorphism. This entails normality of Y as well as the vanishing of the higher derived direct images. A rational singularity is du Bois and Cohen–Macaulay. We refer the reader to [Kem86] for some other aspects of rational singularities. ♢

The following result is a relative of Theorem 3.26.

Corollary 3.33. *In characteristic zero, for connected M_W, the affine cone $\hat{\Lambda}_{W,0}$ has rational singularities and is, in particular, normal.*

Proof. Given a configuration W over a field $\mathbb{K}$ of characteristic zero, lift it to an integer variety. Then for infinitely many prime numbers p the reduction modulo p will be a configuration for M, and by Theorem 3.31 the corresponding $\hat{\Lambda}_{W,0}$ over $\mathbb{Z}/p\mathbb{Z}$ is an F-rational reduction of the model of $\hat{\Lambda}_{W,0}$ over $\mathbb{K}$. This makes $\hat{\Lambda}_{W,0}$ over $\mathbb{K}$ a variety of *F-rational type.* K. Smith and N. Hara proved that F-rational type is equivalent to rational singularities (see [Smi97; TW18; Har98b]). □

Corollary 3.34. *If M is connected, then for any of its configurations W over an F-finite field $\mathbb{K}$, the image of $\hat{\Lambda}_W \setminus \mathrm{V}(u)$ in $\mathbb{P}W \times V^\star \subseteq \mathbb{P}V \times V^\star$ is strongly F-regular.*

Proof. Strong F-regularity descends along field extensions, [Has10, Lem. 3.17]. We may hence assume that $\mathbb{K} = \bar{\mathbb{K}}$.

Away from $\mathrm{V}(u)$, $I_{W,0}$ is the complete intersection of the entries $q_1, \dots, q_r$ of $[Q_W][u]$. But in Gorenstein rings, F-rationality is equivalent to strong F-regularity (see [MP22a, Section 4]). □

Remark 3.35. When $\mathbb{K} = \mathbb{C}$, Corollary 3.34 implies that the variety therein, as well as $\Lambda_W \subseteq \mathbb{P}W \times \mathbb{P}V^\star$, have rational singularities. A concurrent proof appears in [BM26, Cor. 4.1], as a byproduct of a study of the singularities and contact loci of a larger genre of incidence varieties. Such methods cannot recover any of our F-singularity results, since contact loci are less understood in characteristic p. ♢

3.7. **Nash blow-up and projective duality.** In this subsection, we assume that $\mathbb{K}$ is algebraically closed of characteristic zero. Bloch, Esnault and Kreimer [BEK06, §4] hint at an interpretation of X_W as a dual variety. We give a precise formulation of this relation. Our investigation reveals the normalization of the Nash blow-up of X_W to be Λ_W.

First, we recall the general construction of incidence and dual varieties and Nash blow-ups. Let $Z \subseteq \mathbb{P}V$ be a projective variety and consider the projection $p_2 \colon \mathbb{P}V \times \mathbb{P}V^\star \to \mathbb{P}V^\star$. The *projectivized conormal bundle* of its smooth part $Z^{\mathrm{sm}} \subseteq \mathbb{P}V$ is given by

$$\Xi'_Z \coloneqq \{(z,\beta) \in Z^{\mathrm{sm}} \times \mathbb{P}V^\star \mid \beta|_{T_z Z} = 0\}\,.$$

Its Zariski closure in $\mathbb{P}V \times \mathbb{P}V^\star$ is the *incidence variety* of Z,

$$\Xi_Z \coloneqq \overline{\Xi'_Z} \subseteq \mathbb{P}V \times \mathbb{P}V^\star$$

It is equidimensional of dimension

$$\dim(\Xi_Z) = \dim(\Xi'_Z) = n-2 \tag{3.36}$$

with the same number of irreducible components as Z. The *dual variety* of Z is given by (see [GKZ08, Ch. 1, 1.A, 3.A.(1)]),

$$Z^\vee \coloneqq \overline{\{\beta \in \mathbb{P}V^\star \mid \exists z \in Z^{\mathrm{sm}} \colon \beta|_{T_z Z} = 0\}} = p_2(\Xi_Z) \subseteq \mathbb{P}V^\star. \tag{3.37}$$

Identifying $V = V^{\star\star}$, the biduality theorem (see [GKZ08, Ch. 1, Thm. 1.1, (3.1), (3.2)]) states that,

$$\Xi_{Z^\vee} = \Xi_Z, \quad Z^{\vee\vee} = Z. \tag{3.38}$$

A related object is the Nash blow-up of Z, defined whenever Z is equidimensional. From now on, we restrict to the case where Z is a hypersurface. The *Gauss map* sends each smooth point of Z to its tangent space:

$$\eta_Z : Z^{\mathrm{sm}} \to \mathbb{P}V^\star, \quad z \mapsto T_z Z,$$

where we use $T_z Z \subseteq \mathbb{P}V^\star$ as shorthand for the unique $\beta \in \mathbb{P}V^\star$ vanishing on $T_z Z$. The *Nash blow-up* is the Zariski closure in $Z \times \mathbb{P}V^\star$ of its graph $\Gamma_{\eta_Z} = \Xi'_Z$, and agrees with the incidence variety (in the hypersurface case):

$$\Xi_Z = \mathrm{Nash}(Z) \coloneqq \overline{\Gamma_{\eta_Z}} \subseteq Z \times \mathbb{P}V^\star.$$

Returning now to the setting of this paper, write $\overline{w}$ (resp. $\overline{v}$) for the image of $w \in W$ inside $\mathbb{P}W$ (resp. the image of $v \in V$ inside $\mathbb{P}V$). Consider the morphism defined by the Hadamard square

$$\iota_W \colon \mathbb{P}W \to \mathbb{P}V, \quad w = \sum_{j\in E} \ell_j(w) x_j \mapsto \sum_{j \in E} (\ell_j(w))^2 x_j = Q_W(w,w). \tag{3.39}$$

Note that ι_W is finite since it is quasifinite and projective. In [BEK06] this map is assumed to be a closed immersion, which is not true in general.

Example 3.40. Consider the configuration $W \subseteq V$ spanned by the rows of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix}.$$

Then W represents the connected uniform matroid $\mathsf{U}_{3,4}$ but ι_W is not injective. $\diamondsuit$

The differential of ι_W is closely related to $Q_E^\star$ as follows.

Lemma 3.41. *Pick coordinates $z_1, \dots, z_r$ corresponding to a basis $w^1, \dots, w^r$ of W. For $i = 1, \dots, r$, $w \in W$ and $\beta \in V^\star$, we have*

$$\beta(\frac{\partial Q_W(w, w)}{\partial z_i}) = 2Q_W^\star(w, \beta)(w^i).$$

In particular, $(\overline{w}, \beta) \in \Lambda_W$ if $(\iota_W(\overline{w}), \beta) \in \Xi'_{Z_W}$.

Proof. By assumption, $w = \sum_{i=1}^r z_i(w)w^i$ for all $w \in W$. Hence, $w_j = \sum_{i=1}^r z_i(w)w_j^i$ and $\frac{\partial}{\partial z_i} w_j = w_j^i$ for all $j \in \{1, \dots, r\}$. The chain rule thus gives

$$\frac{\partial Q_W(w, w)}{\partial z_i} = \frac{\partial}{\partial z_i} \sum_{j \in E} w_j^2 x_j = \sum_{j \in E} 2w_j w_j^i x_j.$$

Applying $\beta \in V^*$ yields the claimed equality. The particular claim follows since the projective image of $\frac{\partial Q_W(w,w)}{\partial z_i}$ belongs to $T_{\iota_W(w)} Z_W$ if $\iota_W(w) \in Z_W^{\mathrm{sm}}$. $\square$

Since $\mathbb{P}W$ is complete, ι_W is a closed morphism. Its image, denoted by

$$Z_W := \iota_W(\mathbb{P}W) \subseteq \mathbb{P}V,$$

is an irreducible variety with irreducible incidence variety Ξ_{Z_W} projecting to the dual variety $Z_W^\vee$. We show that X_W and Z_W are mutually dual:

Proposition 3.42. *Let $W \subseteq V$ be a connected configuration. Then $(\iota_W \times \mathrm{id})(\Lambda_W) = \Xi_{Z_W}$, $Z_W^\vee = X_W$ and $X_W^\vee = Z_W$. In particular, $\Xi_{Z_W} = \Xi_{X_W} = \mathrm{Nash}(X_W)$.*

Proof. By construction, $\Xi'_{Z_W} \subseteq Z_W^{\mathrm{sm}} \times \mathbb{P}V^\star$ and hence

$$(\iota_W \times \mathrm{id})^{-1}(\Xi'_{Z_W}) \subseteq (\iota_W \times \mathrm{id})^{-1}(Z_W^{\mathrm{sm}} \times \mathbb{P}V^\star) = \iota_W^{-1}(Z_W^{\mathrm{sm}}) \times \mathbb{P}V^\star$$

is a closed subvariety. It is contained in Λ_W by Lemma 3.41. As a consequence

$$(\iota_W \times \mathrm{id})^{-1}(\Xi'_{Z_W}) \subseteq \Lambda_W \cap (\iota_W^{-1}(Z_W^{\mathrm{sm}}) \times \mathbb{P}V^\star) \subseteq \Lambda_W \tag{3.43}$$

is a (nonempty) closed subvariety of an open subset of Λ_W. By Corollary 3.9 and (**3.36**) for $Z = Z_W$, Λ_W and the dense open are irreducible of dimension

$$\dim \Lambda_W = n - 2 = \dim \Xi'_{Z_W}.$$

This equals the dimension of the left hand side of (**3.43**), due to finiteness of ι_W. We conclude that the first inclusion in (**3.43**) is in fact an equality. Since $\iota_W \times$

id is projective, and hence closed, it commutes with taking Zariski closure. Using irreducibility of Ξ_W and Λ_W, it follows that

$$\begin{aligned}\Xi_{Z_W} = \overline{\Xi'_{Z_W}} &= \overline{(\iota_W \times \mathrm{id})\big((\iota_W \times \mathrm{id})^{-1}(\Xi'_{Z_W})\big)} \\ &= \overline{(\iota_W \times \mathrm{id})\big(\Lambda_W \cap (\iota_W^{-1}(Z_W^{\mathrm{sm}}) \times \mathbb{P}V^\star)\big)} \\ &= (\iota_W \times \mathrm{id})\big(\overline{\Lambda_W \cap (\iota_W^{-1}(Z_W^{\mathrm{sm}}) \times \mathbb{P}V^\star)}\big) = (\iota_W \times \mathrm{id})(\Lambda_W).\end{aligned}$$

Using (**3.37**) and Proposition 3.16 we find that

$$Z_W^\vee = p_2(\Xi_{Z_W}) = (p_2 \circ (\iota_W \times \mathrm{id}))(\Lambda_W) = p_2(\Lambda_W) = X_W.$$

By the biduality theorem (**3.38**), then

$$X_W^\vee = Z_W^{\vee\vee} = Z_W, \quad \Xi_{Z_W} = \Xi_{Z_W^\vee} = \Xi_{X_W} = \mathrm{Nash}(X_W). \qquad \square$$

We interpret Λ_W as the normalized Nash blow-up $\widetilde{\mathrm{Nash}(X_W)}$ of X_W, show that both have rational singularities and describe the projection map $\Lambda_W \to X_W$.

Theorem 3.44. *Let $W \subseteq V$ be a connected configuration. Then there is the following commutative diagram of irreducible varieties:*

(**3.45**)
$$\begin{array}{ccccc} & & \widetilde{\mathrm{Nash}(X_W)} & & \\ & \overset{\phi}{\underset{\cong}{\nearrow}} & & \overset{\nu}{\searrow} & \\ \Lambda_W & & \xrightarrow{\quad \iota_W \times \mathrm{id} \quad} & & \mathrm{Nash}(X_W) \\ & \underset{p_2}{\searrow} & & \underset{p_2}{\swarrow} & \\ & & X_W & & \end{array}$$

Both Λ_W and X_W have rational singularities and $Rp_{2,*}\mathscr{O}_{\Lambda_W} = \mathscr{O}_{X_W}$.

Proof. By Proposition 3.42, the morphism $\iota_W \times \mathrm{id}$ in (**3.45**) is defined and surjective. The lower triangle trivially commutes.

By Corollary 3.13, Λ_W is an irreducible normal variety. Since X_W is an irreducible variety, so are $\mathrm{Nash}(X_W)$ and its normalization $\widetilde{\mathrm{Nash}(X_W)}$. By the universal property of normalization (see [GW10, Prop. 12.44]), the morphism $\iota_W \times \mathrm{id}$ in (**3.45**) thus uniquely factors through the normalization morphism ν of $\mathrm{Nash}(X_W)$ as a morphism ϕ. Note that ϕ is quasi-finite, since $\iota_W \times \mathrm{id}$ is so.

By Proposition 3.16 and construction, p_2 on both Λ_W and $\mathrm{Nash}(X_W)$ is an isomorphism over X_W^{sm}, whereas ν is an isomorphism over the normal locus $\mathrm{Nash}(X_W)^{\mathrm{nor}} \supseteq p_2^{-1}(X_W^{\mathrm{sm}})$. It follows that ϕ is an isomorphism over $(p_2 \circ \nu)^{-1}(X_W^{\mathrm{sm}})$ and hence birational. By Zariski's main theorem (see [MO15, §5.6: (N3) $\Longrightarrow$ (N4)]), it follows that ϕ is an isomorphism.

The claims on rational singularities (which is an open condition) follow from Corollary 3.33 and [BW24, Theorems 1.1, 1.2] (later improved in [BMW24, Theorem 1.1]), respectively.

In order to check that $Rp_{2,*}\mathscr{O}_{\Lambda_W} = \mathscr{O}_{X_W}$, pick a resolution of singularities $\tau\colon T \to \Lambda_W$. By Proposition 3.16, then also $p_2 \circ \tau\colon T \to X_W$ is a resolution of singularities. Consider the Grothendieck spectral sequence with second page

$$R^i p_{2,*}(R^j \tau_* \mathscr{O}_T) \implies R^{i+j}(p_2 \circ \tau)_* \mathscr{O}_T.$$

Since both Λ_W and X_W have rational singularities, the spectral sequence simplifies to

$$Rp_{2,*}\mathscr{O}_{\Lambda_W} = R(p_2 \circ \tau)_* \mathscr{O}_T = \mathscr{O}_{X_W}. \qquad \square$$

Example 3.46. For the graphic configuration from Example 3.15, we show that, over $\mathbb{C}$, the birational modification $p_2 : \Lambda_W \to X_W$ made the singularities milder.

For local complete intersections Z embedded in a smooth Y, the singularity invariant par excellence is the minimal exponent $\tilde{\alpha}(Z, Y)$. This positive rational number is computed locally using Bernstein–Sato polynomials. The philosophy is that the larger the value of $\tilde{\alpha}(Z, Y) - \operatorname{codim}_Y(Z)$, the milder the singularities. Note that while minimal exponents depend on the embedding, [Che+24, Prop 4.14] implies that $\tilde{\alpha}(Z, Y) - \operatorname{codim}_Y(Z)$ does not.

Let us compute $\tilde{\alpha}(\Lambda_W, \mathbb{P}W \times \mathbb{P}V^\star)$. Recall that the biprojective variety $\hat{\Lambda}_W \subseteq W \times V^\star$ is cut out by the bihomogeneous defining ideal

$$\Big(\underbrace{(x_1 + x_4 + x_5)u_1 + x_4 u_2 + x_3 u_3}_{=:h_1}, \underbrace{x_4 u_1 + (x_2 + x_4)u_2}_{=:h_2}, \underbrace{x_5 u_1 + (x_3 + x_5)u_3}_{=:h_3}\Big)$$

inside $\mathbb{C}[u_1, u_2, u_3, x_1, x_2, x_3, x_4, x_5]$. Moreover, Λ_W has only two singular points: (α_{124}, β_0) and (α_{135}, β_0). Let $Z = \Lambda_W \setminus V(u_2, x_1) \subseteq \big(\mathbb{P}W \times \mathbb{P}V^\star\big) \setminus V(u_2, x_1) \simeq \operatorname{Spec}(\mathbb{C}[u_1, u_3, x_2, x_3, x_4, x_5]$; let $Z' = \Lambda_W \setminus V(u_3, x_1) \subseteq \big(\mathbb{P}W \times \mathbb{P}V^\star\big) \setminus V(u_3, x_1) \simeq \operatorname{Spec}\mathbb{C}[u_1, u_2, x_2, x_3, x_4, x_5]$. We find that Z is isomorphic to the hypersurface $D_3 \coloneqq (h_3 = 0) \subseteq \operatorname{Spec}\mathbb{C}[u_1, u_3, x_3, x_5]$. There is a formula for the Bernstein–Sato polynomial of a homogeneous polynomial with an isolated singularity; in this case, the minimal exponent is the number of variables divided by the degree. So $\tilde{\alpha}(D_3, \mathbb{C}^4) = 2$ and $\tilde{\alpha}(D_3, \mathbb{C}^4) - \operatorname{codim}_{\mathbb{C}^4}(D_3) = \tilde{\alpha}(Z, \mathbb{C}^6) - \operatorname{codim}_{\mathbb{C}^6}(Z)$, which forces $\tilde{\alpha}(Z, \mathbb{C}^6) = 4$. On the other hand, Z' is isomorphic to the hypersurface $D_2 \coloneqq (h_2 = 0) \subseteq \operatorname{Spec}\mathbb{C}[u_1, u_2, x_2, x_4]$. As before we see that $\tilde{\alpha}(D_2, \mathbb{C}^4) = 2$ and $\tilde{\alpha}(Z', \mathbb{C}^6) = 4$. These two computations on charts stitch together, yielding $\tilde{\alpha}(\Lambda_W, \mathbb{P}W \times \mathbb{P}V^\star) = 4$.

Using Macaulay2 [GS] to compute Bernstein–Sato polynomials on the five canonical affine patches of $\mathbb{P}V^\star$ we see that $\tilde{\alpha}(X_G, \mathbb{P}V^\star) = 3/2$. We conclude the singularities of Λ_W are indeed nicer than those of X_G:

$$1 = \tilde{\alpha}(\Lambda_W, \mathbb{P}W \times \mathbb{P}V^\star) - \operatorname{codim}_{\mathbb{P}W \times \mathbb{P}V^\star}(\Lambda_W) > \tilde{\alpha}(X_G, \mathbb{P}V^\star) - \operatorname{codim}_{\mathbb{P}V^\star}(X_G) = 1/2.$$

We can say even more about the improvement of singularities using the language of higher du Bois (resp. rational) singularities. For the following statements, we use [Sai93, Thm 0.4], [MP22b, Thm F], [Che+24, Thm 1.3], [CDM24, Thm 1.1]. From $\tilde{\alpha}(X_G, \mathbb{P}V^\star) - \operatorname{codim}_{\mathbb{P}V^\star}(X_G) = 1/2$ we conclude that X_G has rational singularities but not the stronger property of 1-du Bois singularities. On the other hand, since $\tilde{\alpha}(\Lambda_W, \mathbb{P}W \times \mathbb{P}V^\star) - \operatorname{codim}_{\mathbb{P}W \times \mathbb{P}V^\star}(\Lambda_W) = 1$, we conclude that Λ_W has 1-du Bois singularities and, in particular, rational singularities.

We will revisit this example in §4.4. $\diamondsuit$

3.8. **Classes and cohomology.** In this subsection, we assume that $\mathbb{K} = \mathbb{C}$. Results from the previous subsection serve to describe various classes and cohomology rings attached to Λ_W in explicit combinatorial terms.

The stratification in Proposition 3.8 yields the class of Λ_W in the ring of varieties, proving Theorem 1.3 from the introduction.

Corollary 3.47. *For any configuration $W \subseteq V$, the class of Λ_W in the Grothendieck ring of varieties over $\mathbb{K}$ equals*

$$\begin{aligned}[\Lambda_W] &= \sum_{\substack{F \in \mathscr{L}_{\mathsf{M}_W} \\ F \neq E}} [\mathbb{P}(W/F)^\circ] \times [\mathbb{P}^{|E| - \operatorname{rank}(\mathsf{M}_W \setminus F) - 1}] \\ &= \sum_{\substack{F \in \mathscr{L}_{\mathsf{M}_W} \\ F \neq E}} \overline{\chi}_{\mathsf{M}_W/F}(\mathbb{L}) \cdot (\mathbb{L}^{|E| - \operatorname{rank}(\mathsf{M}_W \setminus F)} - 1),\end{aligned}$$

where $\mathbb{L} = [\mathbb{A}^1]$ and $\overline{\chi}_{\mathsf{M}}$ denotes the reduced characteristic polynomial of M. □

In contrast, the bidegree of Λ_W depends only on n and r.

Corollary 3.48. *Let $W \subseteq V$ be a configuration of rank $r > 0$. Then the class of Λ_W in the Chow ring of $\mathbb{P}V \times \mathbb{P}V^\star$ is*

$$[\Lambda_W] = [H]^{n-r}([H] + [H^\star])^r,$$

where $H \subseteq V$ and $H^\star \subseteq V^\star$ denote hyperplanes.

Proof. By Corollary 3.9, $\Lambda_W \subseteq \mathbb{P}V \times \mathbb{P}V^\star$ is defined by a regular sequence consisting of $n - r$ generators of bidegree $(1, 0)$ and r generators of bidegree $(1, 1)$. (See, e.g., [Ful98, Ex. 8.4.2]). The claimed equality follows. □

Bloch showed that the mixed Hodge structure on the cohomology of Λ_W is mixed Tate (see [Blo20, Prop. 4.1]), that is, $H^{2k}(\Lambda_W, \mathbb{C})$ is pure of weight k for all $k \geq 0$, and odd-dimensional cohomology is zero. In the round case, the cohomology ring is determined combinatorially by n and r. Additively, this is clear, since a projective bundle over projective space has the Betti numbers of a product. As an algebra, we have:

Proposition 3.49. *For any round configuration $W \subseteq V$ of rank r, the Chow ring and cohomology ring of Λ_W are given by*

$$A^\bullet(\Lambda_W) = H^{2\bullet}(\Lambda_W, \mathbb{Z}) \cong \mathbb{Z}[a, b]/\langle a^r, b^{-r}(b-a)^n \rangle,$$

where $\deg(a) = \deg(b) = 1$, and (formally)

$$b^{-r}(b-a)^n = \sum_{k=0}^{r} \binom{n}{k} (-a)^k b^{n-r-k}.$$

Proof. The cycle class map between $A^\bullet$ and $H^{2\bullet}$ is an isomorphism [Ful98, Ex. 19.1.11(d)], so let us compute the Chow ring. By Propositions 3.4 and 3.8, $\Lambda_W \cong \operatorname{Proj}\operatorname{Sym}\mathscr{F}$ is a $\mathbb{P}^{n-r-1}$-bundle over $\mathbb{P}W$. Using [Ful98, Ex. 8.3.4],

$$A^\bullet(\Lambda_W, \mathbb{Z}) \cong A^\bullet(\mathbb{P}W)[b]/(b^{n-r} + c_1(\mathscr{F}^\vee)b^{n-r-1} + \cdots + c_{n-r}(\mathscr{F}^\vee)),$$

where $b := \mathscr{O}_{\Lambda_W}(1)$ is the pullback of $[H^\star] - [H]$. The presentation (**3.3**) of $\mathscr{F}$ yields the total Chern class

$$\sum_{k=0}^{n} c_k(\mathscr{F}^\vee) = c(\mathscr{F}^\vee) = c(\mathscr{O}_{\mathbb{P}W}(-1)^n)/c(\mathscr{O}_{\mathbb{P}W}^r) = (1-a)^n = \sum_{k=0}^{n} \binom{n}{k}(-a)^k,$$

where $a := [H] \in A^\bullet(\mathbb{P}W)$. The claimed isomorphism follows. □

4. Tropical resolutions

Our goal now is to construct an explicit resolution of singularities for the configuration hypersurface X_W for general configurations W using tropical techniques. In this section, we assume $\mathbb{K} = \mathbb{C}$, and that configurations W are connected. Then X_W and Λ_W are irreducible varieties and can be considered as the closure of their torus parts.

4.1. **Restriction to tori.** Let $\mathbb{T}_{E,E^\star} := \mathbb{T}_E \times \mathbb{T}_{E^\star}$ and $\hat{\mathbb{T}}_{E,E^\star} := \hat{\mathbb{T}}_E \times \hat{\mathbb{T}}_{E^\star}$ denote the respective tori in $\mathbb{P}V \times \mathbb{P}V^\star$ and $V \times V^\star$. Consider the torus parts of Λ_W and $\hat{\Lambda}_W$,

$$\Lambda_W^\circ := \Lambda_W \cap \mathbb{T}_{E,E^\star} \quad \text{and} \quad \hat{\Lambda}_W^\circ := \hat{\Lambda}_W \cap \hat{\mathbb{T}}_{E,E^\star}.$$

The graph of the action of $\hat{\mathbb{T}}_E$ on $\hat{\mathbb{T}}_{E^\star}$ defines an automorphism

$$\hat{m}\colon \hat{\mathbb{T}}_{E,E^\star} \to \hat{\mathbb{T}}_{E,E^\star}, \quad (u,\beta) \mapsto (u, u\beta),$$

equivariant with respect to the subtorus defining $\mathbb{T}_{E,E^\star}$. It induces an automorphism

$$m\colon \mathbb{T}_{E,E^\star} \to \mathbb{T}_{E,E^\star}, \quad (u,\beta) \mapsto (u, u\beta). \tag{4.1}$$

Remark 4.2. The automorphism $\hat{m}$ of $\hat{\mathbb{T}}_{E,E^\star}$ corresponds to a bihomogeneous, monomial $\mathbb{K}$-algebra automorphism of the coordinate ring:

$$\hat{m}^\sharp\colon \mathbb{K}[\hat{\mathbb{T}}_{E,E^\star}] \to \mathbb{K}[\hat{\mathbb{T}}_{E,E^\star}], \quad y_i \mapsto y_i, \quad x_i \mapsto x_i/y_i.$$

This follows from

$$(u\beta)(x_i) = \beta(u^{-1}x_i) = \beta(y_i(u)^{-1}x_i) = y_i(u)^{-1}\beta(x_i) = (x_i/y_i)(u,\beta).$$

Then the automorphism m of $\mathbb{T}_{E,E^\star}$ corresponds to a monomial $\mathbb{K}$-algebra automorphism of the coordinate ring:

$$m^\sharp\colon \mathbb{K}[\mathbb{T}_{E,E^\star}] \to \mathbb{K}[\mathbb{T}_{E,E^\star}], \quad y_i/y_k \mapsto y_i/y_k, \quad x_i \mapsto (x_i/x_k)/(y_i/y_k). \qquad \diamondsuit$$

Notation 4.3. For any configuration $W \subseteq V$, we denote by

$$\Lambda_W^{\mathrm{irr}} := \overline{\Lambda_W^\circ} \subseteq \mathbb{P}V \times \mathbb{P}V^\star$$

the closure of Λ_W° in $\mathbb{P}V \times \mathbb{P}V^\star$. $\diamondsuit$

Proposition 4.4. *For any configuration $W \subseteq V$, m and $\hat{m}$ induce isomorphisms*

$$m\colon (\mathbb{P}W)^\circ \times (\mathbb{P}W^\perp)^\circ \to \Lambda_W^\circ, \quad \hat{m}\colon W^\circ \times (W^\perp)^\circ \to \hat{\Lambda}_W^\circ.$$

If M_W is a connected matroid then $\Lambda_W^\circ \subseteq \mathbb{T}_{E,E^\star}$ is a smooth very affine variety with closure $\Lambda_W^{\mathrm{irr}} = \Lambda_W$.

Proof. It suffices to prove the first claim for $\hat{m}$. By Definition 3.1 and Lemma 2.9, we have $(w, \beta) \in \hat{\Lambda}_W^\circ = \hat{\Lambda}_W \cap (\hat{\mathbb{T}}_E \times \hat{\mathbb{T}}_{E^\star})$ if and only if $w \in W \cap \hat{\mathbb{T}}_E$ and $\beta \in \hat{\mathbb{T}}_{E^\star}$ such that $0 = Q_W^\star(w, \beta) = (w^{-1}\beta)|_W$. Equivalently, $(w, w^{-1}\beta) \in (W \cap \hat{\mathbb{T}}_E) \times (W^\perp \cap \hat{\mathbb{T}}_{E^\star})$, and by definition, $\hat{m}(w, w^{-1}\beta) = (w, \beta)$.

Since we assume $n > r > 0$, because M_W is connected, it has an edge that is neither loop or coloop. Then $\mathbb{P}W^\circ$ and $(\mathbb{P}W^\perp)^\circ$ are nonempty and smooth. The image Λ_W° of m is thus dense open in Λ_W, and smooth due to the first claim. By Corollary 3.9, Λ_W is irreducible. Then also Λ_W° is irreducible with closure $\overline{\Lambda_W^\circ} = \Lambda_W$. □

Corollary 4.5. *Let $W \subseteq V$ be a connected configuration. Then there is a commutative square of birational maps of irreducible varieties*

$$\begin{array}{ccc} \Lambda_W^\circ & \hookrightarrow & \Lambda_W \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} \\ X_W^\circ & \hookrightarrow & X_W. \end{array}$$

Proof. Commutativity holds trivially. By Propositions 3.16 and 4.4, the right projection and the upper inclusion are birational morphisms of irreducible varieties. Then the same holds for the remaining morphisms. □

In §4.3 we will replace $\mathbb{P}V \times \mathbb{P}V^\star$ in the definition of Λ_W^{irr} by a finer $\mathbb{T}_{E,E^\star}$-toric variety in order to obtain a smooth model of the configuration hypersurface X_W for general configurations W.

Proposition 4.6. *For any configuration $W \subseteq V$, the biprojective coordinate ring of Λ_W^{irr} is isomorphic to the $\mathbb{K}$-subalgebra*

$$L_W := \mathbb{K}[\ell_1, \ldots, \ell_n, \ell_1^\perp/\ell_1, \ldots, \ell_n^\perp/\ell_n] \subset \mathbb{K}(W \times W^\perp).$$

Proof. Denote the respective coordinate rings of $V \times V^\star$, $\hat{\mathbb{T}}_{E,E^\star}$ and $W^\circ \times (W^\perp)^\circ$ by

$$\begin{aligned} R &:= \mathbb{K}[V \times V^\star] \cong \mathbb{K}[y_1, \ldots, y_n, x_1, \ldots, x_n], \\ S &:= \mathbb{K}[\hat{\mathbb{T}}_{E,E^\star}] \cong \mathbb{K}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}, x_1^{\pm 1} \ldots, x_n^{\pm 1}], \\ T &:= \mathbb{K}[W^\circ \times (W^\perp)^\circ] \cong \mathbb{K}[\ell_1^{\pm 1}, \ldots, \ell_n^{\pm 1}, (\ell_1^\perp)^{\pm 1} \ldots, (\ell_n^\perp)^{\pm 1}]. \end{aligned}$$

Composing the automorphism $\hat{m}^\sharp$ from Remark 4.2 with the inclusion $R \hookrightarrow S$ and the restriction to $W^\circ \times (W^\perp)^\circ$ yields a map of $\mathbb{K}$-algebras

$$\phi\colon R \hookrightarrow S \xrightarrow[\cong]{\hat{m}^\sharp} S \twoheadrightarrow T, \quad y_i \mapsto \ell_i, \quad x_i \mapsto \ell_i^\perp/\ell_i,$$

with image L_W. Denote by $I := \ker\phi$ its kernel. The induced isomorphism $\overline{\phi}$ fits into a commutative square of $\mathbb{K}$-algebras

$$\begin{array}{ccc} R/I & \xrightarrow[\cong]{\bar\phi} & L_W \\ \big\downarrow & & \big\downarrow \\ S/SI & \hookrightarrow & T. \end{array}$$

By construction, the bottom map induces the map of coordinate rings associated with the isomorphism

$$\hat{m}\colon W^\circ \times (W^\perp)^\circ \to \hat{\Lambda}^\circ_W$$

in Proposition 4.4. Since it is injective and maps onto T, it is isomorphism. Then SI is the ideal of $\hat{\Lambda}^\circ_W \subseteq \hat{\mathbb{T}}_{E,E^\star}$ and pulls back to the ideal $R \cap SI = I$ of $\hat{\Lambda}^\circ_W \subseteq V \times V^\star$. Since $\hat{\Lambda}^\circ_W$ is a biconical subset, I agrees with the ideal of $\Lambda^\circ_W \subseteq \mathbb{P}V \times \mathbb{P}V^\star$, defining its closure Λ^{irr}_W. This turns R/I into the biprojective coordinate ring of Λ^{irr}_W, and $\bar\phi$ becomes the desired isomorphism. ☐

4.2. **Fans and tropicalization.** Tropical geometry gives a framework for studying toric closures of very affine varieties, such as Λ^{irr}_W, by means of fans. For a very affine variety $Y \subseteq \mathbb{T}_E$, the tropicalization $\mathrm{trop}(Y)$ (with respect to the trivial valuation on the field) is the underlying set of a rational, polyhedral fan, together with integer weights on the maximal cones. The fan structure on the tropicalization is not unique, but by choosing a sufficently fine structure one may arrange it to be smooth. We use Proposition 4.4 to describe the tropicalization of Λ°_W in terms of *Bergman fans*. We begin by recalling the relevant definitions.

Consider $e_i := x_i$, $i \in E$, as the unit vectors of a lattice $\mathbb{Z}^E \subseteq V$. For each subset $S \subseteq E$, we write an indicator vector $e_S := \sum_{i\in S} e_i$. We denote the co-character lattice of $\mathbb{T}_E$ and its associated real vector space by

$$N_{E,\mathbb{Z}} := \mathrm{Hom}(\mathbb{K}^\times, \mathbb{T}_E) \cong \mathbb{Z}^E/\mathbb{Z}e_E \quad \text{and} \quad N_E := N_{E,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^E/\mathbb{R}e_E.$$

Negation is an automorphism defined over $\mathbb{Z}$

$$-\colon N_E \to N_E$$

sending $e_S \mapsto e_{E\setminus S}$ for all $S \subseteq E$. Define $e_i^\star$, $N^\star_{E,\mathbb{Z}}$ and $N^\star_E$ using $\mathbb{T}_{E^\star}$ and y_i accordingly. We identify the coordinate tori $\hat{\mathbb{T}}_E$ and $\hat{\mathbb{T}}_{E^\star}$ by the requirement $x_iy_i = 1$. The resulting identification of co-character lattices then reads

$$N^\star_{E,\mathbb{Z}} = N_{E,\mathbb{Z}}, \quad e_i^\star = -e_i. \tag{4.7}$$

The purpose of this identification will become clear in the following Lemma 4.9 and Proposition 4.11. We write

$$N_{(E,E),\mathbb{Z}} := N_{E,\mathbb{Z}} \oplus N_{E,\mathbb{Z}} \quad \text{and} \quad N_{E,E} := N_{(E,E),\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R} = N_E \oplus N_E$$

for the the co-character lattices and associated real vector spaces of both $\mathbb{T}_E \times \mathbb{T}_E$ and $\mathbb{T}_{E,E^\star}$. The indicator vectors in the two summands of $N_{(E,E),\mathbb{Z}}$ will be denoted by e_S and f_S, respectively. We consider the maps

$$\pi_1, \pi_2\colon N_{E,E} \to N_E \quad \text{and} \quad \delta\colon N_{E,E} \to N_E$$

defined over $\mathbb{Z}$, the two projection maps π_1 and π_2 given by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$, respectively, and the addition map δ given by $\delta(x,y) = x + y$.

For a rational, polyhedral fan Σ, we denote by $|\Sigma|$ its support and by $\mathbb{P}(\Sigma)$ its toric variety. The simplicial, unimodular fans are called smooth because they correspond to smooth toric varieties.

Example 4.8. We collect the examples of fans that we will relevant for us.

(a) The fan Γ_E is the smooth complete fan in N_E whose cones are spanned by all proper subsets of the coordinate vectors $\{e_i \mid i \in E\}$. The associated toric variety $\mathbb{P}(\Gamma_E) \cong \mathbb{P}V$ is the projective space of V and $\mathbb{P}(-\Gamma_E) \cong \mathbb{P}V^\star$ its dual by (4.7).

(b) The *permutohedral fan* Σ_E is a smooth complete fan in N_E. Its rays are spanned by indicator vectors e_S for which $S \subseteq E$ is a nonempty, proper subset. The cones

$$\sigma_{\mathbf{S}} \coloneqq \sum_{i=1}^{k} \mathbb{R}_{\geq 0} \cdot e_{S_i} \subseteq N_E$$

are indexed by strict flags of k distinct, nonempty proper subsets

$$\mathbf{S} = (\varnothing \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq E),$$

where $k \geq 0$. Since $-e_S = e_{E \setminus S}$, Σ_E is stable under negation, that is, $-\Sigma_E = \Sigma_E$.

(c) For a matroid M on E without loops, the *Bergman fan* Σ_{M} (see [AK06; FS05]) is an induced and hence smooth subfan of Σ_E of dimension $\operatorname{rank} \mathsf{M} - 1$. Its rays are spanned by indicator vectors e_F for which F is a nonempty, proper flat of M. The same holds for the subfan $-\Sigma_{\mathsf{M}}$ of $-\Sigma_E = \Sigma_E$. In particular, $-\Sigma_{\mathsf{M}} \times \Sigma_{\mathsf{M}^\perp}$ is a (smooth) induced subfan of $\Sigma_E \times \Sigma_E$.

(d) The *bipermutohedral fan* $\Sigma_{E,E}$ is a smooth, complete fan in $N_{E,E}$, which refines the product fan $\Sigma_E \times \Sigma_E$ and admits maps of fans (see [ADH23, Props. 2.2, 2.11])

$$\pi_1, \pi_2 \colon \Sigma_{E,E} \to \Sigma_E \quad \text{and} \quad \delta \colon \Sigma_{E,E} \to \Gamma_E.$$

Its rays are generated by vectors $e_S + f_T$ indexed by *bisubsets* $S|T$. These are, by definition, pairs of nonempty sets $S, T \subseteq E$ for which $S \cup T = E$ and $S \cap T \neq E$. Its cones

$$\sigma_{\mathbf{S}|\mathbf{T}} \coloneqq \sum_{i=1}^{k} \mathbb{R}_{\geq 0} \cdot e_{S_i|T_i} \subseteq N_{E,E}$$

are indexed by all *biflags* $\mathbf{S}|\mathbf{T}$. These are pairs of flags

$$\mathbf{S} = (S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k) \quad \text{and} \quad \mathbf{T} = (T_1 \supseteq T_2 \supseteq \cdots \supseteq T_k),$$

considered as collections of bisubsets $\{S_i|T_i \mid 1 \leq i \leq k\}$, for which $\bigcup_{i=1}^k (S_i \cap T_i) \neq E$.

$\diamondsuit$

Lemma 4.9. *The tropicalization of the automorphism* (4.1) *is a lattice automorphism*

$$\mu \coloneqq \operatorname{trop}(m) = \operatorname{id}_{N_E} \oplus \delta \colon N_{E,E} \to N_{E,E}$$

which satisfies $\pi_2 \circ \mu = \delta$.

Proof. According to Remark 4.2, the automorphism $m^\sharp$ corresponding to (**4.1**) on coordinate rings is a monomial map defined on exponent vectors by $(x, y) \mapsto (x, y-x)$. The contragredient action from (**4.7**) negates the second coordinate, giving the map (see [MS15, p. 87])

$$(x, y) = (x, -(-y)) \mapsto (x, x-(-y)) = (x, x+y). \qquad \square$$

Notation 4.10. For any loopless matroid M on E, consider the smooth fan in $N_{E,E}$

$$\Delta_{\mathsf{M}} := -\mu((-\Sigma_{\mathsf{M}}) \times \Sigma_{\mathsf{M}^\perp}). \qquad \diamondsuit$$

Proposition 4.11. *Let W be a configuration whose matroid $\mathsf{M} = \mathsf{M}_W$ has no loops or coloops. Then $\operatorname{trop}(\Lambda_W^\circ) = |\Delta_{\mathsf{M}}|$ with all weights equal to* 1.

Proof. By Proposition 4.4 and Lemma 4.9, we have (see [MS15, Cor. 3.2.13, (5.5.7)])

$$\begin{aligned} \operatorname{trop}(\Lambda_W^\circ) &= \operatorname{trop}(m)\big(\operatorname{trop}((\mathbb{P}W)^\circ \times (\mathbb{P}W^\perp)^\circ)\big) \\ &= \mu\big(\operatorname{trop}((\mathbb{P}W)^\circ) \times \operatorname{trop}((\mathbb{P}W^\perp)^\circ)\big). \end{aligned}$$

The set $\operatorname{trop}((\mathbb{P}W)^\circ) \subseteq N_E$ is the support of the Bergman fan Σ_{M} (see [FS05, Lem. 6.5], [AK06, Thm. 1]). Similarly, using (**4.7**), we identify $\operatorname{trop}((\mathbb{P}W^\perp)^\circ) \subseteq N_E^\star$ with the support of the fan $-\Sigma_{\mathsf{M}^\perp}$ in N_E. The weights on Bergman fans all equal 1 (see [AHK18, Prop. 5.2]). This property is preserved under products of fans (see [GKM09, Ex. 2.9.(iii)]) and under lattice automorphisms. The claim follows. $\square$

4.3. **Tropical compactifications.** We recall the following notions introduced by Tevelev [Tev07].

Definition 4.12. A compactification $\overline{Y}$ of a very affine variety $Y \subseteq \mathbb{T}$ in a $\mathbb{T}$-toric variety $\mathbb{P}$ is called *tropical/schön* if $\overline{Y}$ is proper and the torus multiplication map $\mathbb{T} \times \overline{Y} \to \mathbb{P}$ is surjective and flat/smooth. If such a compactification exists, then also $Y \subseteq \mathbb{T}$ is called *tropical/schön.* $\diamondsuit$

Remark 4.13. Hacking [Hac08, Lem. 2.7] observed that $\overline{Y}$ being schön is equivalent to smoothness of $\overline{Y} \cap O$ for all orbits O in $\mathbb{P}$. In particular, $Y = \overline{Y} \cap \mathbb{T}$ must be smooth in order for $\overline{Y}$ to be schön. $\diamondsuit$

Tevelev proved the following fundamental results.

Theorem 4.14 ([Tev07, Thms. 1.2, 1.4, Prop. 2.5])**.** *Any very affine variety $Y \subseteq \mathbb{T}$ tropically compactifies in some $\mathbb{T}$-toric variety $\mathbb{P}$. If it does so in $\mathbb{P} = \mathbb{P}(\Sigma)$ for some rational, polyhedral fan Σ, then $|\Sigma| = \operatorname{trop}(Y)$, and it also does so in $\mathbb{P}(\Sigma')$ for any refinement Σ' of Σ. In particular, there is always a tropical compactification in a smooth $\mathbb{P}$. If one tropical compactification of Y is schön, then all are.* $\square$

Our starting point is the following observation.

Proposition 4.15. *For any configuration $W \subseteq V$, the very affine variety $\Lambda_W^\circ \subseteq \mathbb{T}_{E,E^\star}$ is schön.*

Proof. Any hyperplane arrangement complement $(\mathbb{P}W)^\circ \subseteq \mathbb{T}_E$ is schön due to [Tev07, Thm. 1.5]. By Proposition 4.4, $\Lambda_W^\circ \subseteq \mathbb{T}_{E,E^\star}$ is isomorphic to a product of such under the torus automorphism in (**4.1**). Schön compactifications are preserved under products and torus automorphisms. The claim follows. $\square$

A key result due to Luxton and Qu makes it easy to find schön compactifications.

Theorem 4.16 ([LQ11, Thm. 1.5])**.** *Let $Y \subseteq \mathbb{T}$ be a schön very affine variety. Then any rational, polyhedral fan Σ with $|\Sigma| = \operatorname{trop}(Y)$ gives rise to a schön compactification $\overline{Y}$ in $\mathbb{P}(\Sigma)$.* $\square$

For our purpose, the following well-known consequence is useful.

Corollary 4.17. *Let $Y \subseteq \mathbb{T}$ be a schön very affine variety. Then any smooth rational, polyhedral fan Σ with $|\Sigma| = \operatorname{trop}(Y)$ gives rise to a smooth schön compactification $\overline{Y}$ in $\mathbb{P}(\Sigma)$.*

Proof. By hypothesis and Theorem 4.16, both $\mathbb{P} := \mathbb{P}(\Sigma) \to \operatorname{Spec}\mathbb{K}$ and $\mathbb{T} \times \overline{Y} \to \mathbb{P}$ are smooth, hence so is the composite $\mathbb{T} \times \overline{Y} \to \operatorname{Spec}\mathbb{K}$. Base change of the smooth map $\mathbb{T} \to \operatorname{Spec}\mathbb{K}$ along $\overline{Y} \to \operatorname{Spec}\mathbb{K}$ shows that $\mathbb{T} \times \overline{Y} \to \overline{Y}$ is surjective and smooth. Then it follows from the diagram

$$\begin{array}{ccc} \mathbb{T} \times \overline{Y} & \longrightarrow & \overline{Y} \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec}\mathbb{K} & \end{array}$$

that $\overline{Y} \to \operatorname{Spec}\mathbb{K}$ is too (see [Sta23, Lemma 02K5]). $\square$

In order to obtain a smooth model of the configuration hypersurface X_W for general configurations W, we now replace $\mathbb{P}V \times \mathbb{P}V^\star$ in the definition of Λ_W^{irr} with a general $\mathbb{T}_{E,E^\star}$-toric variety (see §4.1).

Notation 4.18. For any configuration $W \subseteq V$ and any rational, polyhedral fan Σ in $N_{E,E}$, we denote by

$$\Lambda_W(\Sigma) := \Lambda_W(\mathbb{P}(\Sigma)) := \overline{\Lambda_W^\circ} \subseteq \mathbb{P}(\Sigma)$$

the irreducible subvariety obtained as the closure of Λ_W° in the $\mathbb{T}_{E,E^\star}$-toric variety $\mathbb{P}(\Sigma)$ (see Proposition 4.4). $\diamondsuit$

Remark 4.19. For any connected configuration $W \subseteq V$, we have

$$\Lambda_W = \Lambda_W(\mathbb{P}V \times \mathbb{P}V^\star) = \Lambda_W(\Gamma_E \times (-\Gamma_E)),$$

due to Proposition 4.4 (see Example 4.8.(a)). $\diamondsuit$

We are interested in fans Σ for which $\Lambda_W(\Sigma)$ is not only smooth, but also maps to X_W. The correct notion is the following.

Definition 4.20. We will say that a smooth fan Δ in $N_{E,E}$ defines a *tropical resolution* for a matroid M, or for a configuration $W \subseteq V$ with matroid $\mathsf{M} = \mathsf{M}_W$, if $|\Delta| = |\Delta_{\mathsf{M}}|$ and the projection $-\pi_2\colon N_{E,E} \to N_E$ defines a map of fans $\Delta \to \Gamma_E$. We call it *biprojective*, if also π_1 defines such a map of fans. Equivalently, this means that the identity of $N_{E,E}$ defines a map of fans $\Delta \to \Gamma_E \times (-\Gamma_E)$. $\diamondsuit$

We note that the smooth fan Δ_{M} itself is not in general a tropical resolution, because it fails the condition on π_2. In §4.4, we rectify the defect by producing a suitable refinement.

Theorem 4.21. *Let $W \subseteq V$ be a connected configuration. If Δ defines a tropical resolution for W, then $\Lambda_W(\Delta)$ is a smooth schön compactification of $\Lambda_W^\circ \subseteq \mathbb{T}_{E,E^\star}$ and $\pi_2 \colon N_{E,E} \to N_E$ induces a birational surjection $p \colon \Lambda_W(\Delta) \to X_W$.*

Proof. By Propositions 4.11, 4.15 and Corollary 4.17, $\Lambda_W^\circ \subseteq \mathbb{T}_{E,E^\star}$ is schön, and the smooth $\mathbb{T}_{E,E^\star}$-toric variety $\mathbb{P}(\Delta)$ makes $\Lambda_W(\Delta)$ a smooth tropical compactification.

By hypothesis, π_2 defines a toric morphism $p \colon \mathbb{P}(\Delta) \to \mathbb{P}V^\star$ (see Example 4.8.(a) and [CLS11, Thm. 3.4.11]), which fits into a commutative diagram:

$$\begin{array}{ccccc} \Lambda_W^\circ & \hookrightarrow & \Lambda_W(\Delta) & \hookrightarrow & \mathbb{P}(\Delta) \\ \downarrow & & \downarrow & & \downarrow p \\ X_W^\circ & \hookrightarrow & X_W & \hookrightarrow & \mathbb{P}V^\star \end{array}$$

Each row contains a very affine variety (left) and its closure (middle) in a toric variety (right) (see Proposition 2.13). The left vertical map is birational (see Corollary 4.5). Since p is continuous, it induces the dashed map, which is closed because $\Lambda_W(\Delta)$ is proper. The image of $\Lambda_W(\Delta)$ is then closed in X_W, and contains a dense open subset. It follows that the dashed map is a birational surjection. $\square$

The construction of $\Lambda_W(\Sigma)$ is functorial in the following sense.

Proposition 4.22. *Let $W \subseteq V$ be a connected configuration. Suppose that the identity of $N_{E,E}$ defines a map $\Sigma \to \Sigma'$ of rational, polyhedral fans such that π_2 defines a map of fans $\Sigma' \to -\Gamma_E$. Then there is an induced commutative diagram:*

$$\begin{array}{ccc} \Lambda_W(\Sigma) & \xrightarrow{\;f\;} & \Lambda_W(\Sigma') \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X_W & \end{array}$$

If $|\Delta_{\mathsf{M}}| \subseteq |\Sigma|$, then f and p are birational surjections.

Proof. We argue as in the proof of Theorem 4.21. Since f is the identity on Λ_W°, the maps exist by continuity of the induced toric morphisms, and $p' \circ f = p$.

By a result of Tevelev (see [Tev07, Prop. 2.3]), the condition on supports $\operatorname{trop}(\Lambda_W^\circ) = |\Delta_{\mathsf{M}}| \subseteq |\Sigma|$ (see Proposition 4.11) makes $\Lambda_W(\Sigma)$ proper and thus f and p surjective. $\square$

Corollary 4.23. *Let $W \subseteq V$ be a connected configuration. If Δ defines a biprojective tropical resolution for W, then the morphism $p \colon \Lambda_W(\Delta) \to X_W$ induced by π_2 factors through $p_2 \colon \Lambda_W \to X_W$.*

Proof. Apply Proposition 4.22 to $\Sigma = \Delta$ and $\Sigma' = \Gamma_E \times (-\Gamma_E)$ (see Remark 4.19). The required map of fans is given by definition of a biprojective tropical resolution. $\square$

4.4. **Square conormal fans.** Now we give a combinatorial recipe to construct a tropical resolution for any matroid. Our construction uses the bipermutohedral fan (see Example 4.8.(d)).

Definition 4.24. For a matroid M on E without loops or coloops, we define the *square conormal fan* of M as the (cone-wise) intersection of fans

$$\Sigma_{-\mathsf{M},\mathsf{M}^\perp} := ((-\Sigma_{\mathsf{M}}) \times \Sigma_{\mathsf{M}^\perp}) \cap \Sigma_{E,E}.$$

$\diamondsuit$

Remark 4.25. The definition of the square conormal fan closely resembles that of the *conormal fan* $\Sigma_{\mathsf{M},\mathsf{M}^\perp} = (\Sigma_{\mathsf{M}} \times \Sigma_{\mathsf{M}^\perp}) \cap \Sigma_{E,E}$, which plays an important role in [ADH23]; however, these two fans are not isomorphic. They are tropicalizations of incidence varieties coming from the Hadamard square immersion (**3.39**) in the former case, and from the logarithmic immersion of $\mathbb{P}W$ given by $x \mapsto \log|x|$ in the latter. $\diamondsuit$

Definition 4.26. By a *square biflat* $F \subseteq G$ of a matroid M on E we mean a pair (F, G) of flats $F \in \mathscr{L}_{\mathsf{M}}$ and $G \in \mathscr{L}_{\mathsf{M}^\perp}$, such that and $(E \backslash F)|G$ is a bisubset: that is, $F \subseteq G$, and if $F = \varnothing$, then $\varnothing \neq G \subsetneq E$. $\diamondsuit$

Proposition 4.27. *The square conormal fan $\Sigma_{-\mathsf{M},\mathsf{M}^\perp}$ is the induced subfan of $\Sigma_{E,E}$ whose rays are indexed by square biflats $F \subseteq G$. In particular, $\Sigma_{-\mathsf{M},\mathsf{M}^\perp}$ is smooth.*

Proof. The fan $-\Sigma_{\mathsf{M}}$ is an induced subfan of Σ_E (see Example 4.8.(c)). Combined with the identity $-e_F = e_{E\backslash F}$ for $F \in \mathscr{L}_{\mathsf{M}}$, the following Lemma 4.28 yields the claim. $\square$

The next lemma makes use of the fact that any nonzero incidence vector e_S in $|\Sigma_E|$ spans a ray of Σ_E.

Lemma 4.28. *For any two induced subfans Σ_1 and Σ_2 of Σ_E, $(\Sigma_1 \times \Sigma_2) \cap \Sigma_{E,E}$ is the subfan of $\Sigma_{E,E}$ induced by the rays indexed by bisubsets $S|T$ for which $e_S \in |\Sigma_1|$ and $f_S \in |\Sigma_2|$.*

Proof. By hypothesis, $\Sigma_1 \times \Sigma_2$ is an induced subfan of $\Sigma_E \times \Sigma_E$. In particular, $\Sigma_1 \times \Sigma_2$ consists of cones of $\Sigma_E \times \Sigma_E$, and the latter is refined by $\Sigma_{E,E}$. Each cone of $(\Sigma_1 \times \Sigma_2) \cap \Sigma_{E,E}$ is thus a cone of $\Sigma_{E,E}$.

Suppose that $S|T$ is a bisubset such that $e_S + f_T$ spans a ray in $|\Sigma_1 \times \Sigma_2| = |\Sigma_1| \times |\Sigma_2|$. Then e_S lies in a cone $\sigma_{\mathbf{S}}$ of Σ_1 as in Example 4.8.(c). Since Σ_E is smooth, $e_S \in \langle e_{S_1}, \dots, e_{S_k} \rangle_{\mathbb{N}}$ and hence $S \in \{\varnothing, S_1, \dots, S_k\}$. Thus, e_S is zero or spans a ray of Σ_1.

Let now $\sigma = \sigma_{\mathbf{S}|\mathbf{T}}$ be any cone of $\Sigma_{E,E}$ as in Example 4.8.(d) with all its rays in $|\Sigma_1 \times \Sigma_2|$. Then $e_{S_1}, \dots, e_{S_k}$ span a cone σ_1 of Σ_E. By the above, it is a cone of the induced subfan Σ_1. With a similar cone σ_2 of Σ_2, the cone $\sigma_1 \times \sigma_2$ of $\Sigma_1 \times \Sigma_2$ then contains σ. This makes σ a cone of $(\Sigma_1 \times \Sigma_2) \cap \Sigma_{E,E}$ and the claim follows. $\square$

Corollary 4.29. *The cones $\sigma_{\mathbf{F}|\mathbf{G}}$ of $\Sigma_{-\mathsf{M},\mathsf{M}^\perp}$ are indexed by biflags*

$$\mathbf{F}|\mathbf{G} = \{(E \backslash F_i, G_i) \mid 1 \leq i \leq k\}$$

where $F_i \in \mathscr{L}_{\mathsf{M}}$ and $G_i \in \mathscr{L}_{\mathsf{M}^\perp}$. Explicitly, this means that

$$\begin{array}{ccccccccccc} E & \supsetneq & F_1 & \supseteq & F_2 & \supseteq & \cdots & \supseteq & F_k & & \\ & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} & & \\ & & G_1 & \supseteq & G_2 & \supseteq & \cdots & \supseteq & G_k & \supsetneq & \varnothing \end{array} \qquad \text{and} \quad \bigcup_{i=1}^{k} (_i \setminus F_i \neq E.$$

Proof. Using the identity $-e_F = e_{E\setminus F}$, this follows from Proposition 4.27 and a reformulation of the conditions on biflags in Example 4.8(d). □

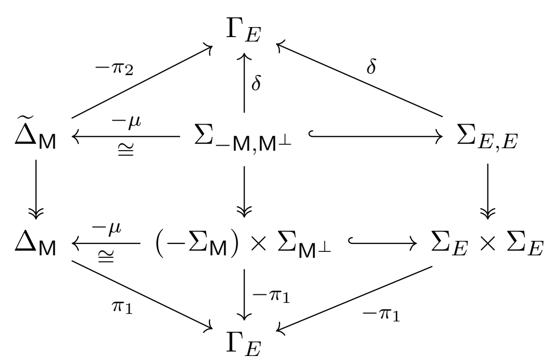
Notation 4.30. For any loopless matroid M on E, consider the smooth fan in $N_{E,E}$

$$\widetilde{\Delta}_{\mathsf{M}} := -\mu(\Sigma_{-\mathsf{M},\mathsf{M}^\perp}). \qquad \diamondsuit$$

Up to isomorphism, then, the square conormal fan refines Δ_{M}.

Theorem 4.31. *The fan $\widetilde{\Delta}_{\mathsf{M}}$ defines a biprojective tropical resolution for M.*

Proof. We need to check that $\pi_1 \times \pi_2 \colon \widetilde{\Delta}_{\mathsf{M}} \to \Gamma_E \times (-\Gamma_E)$ is in fact a map of fans. This can be seen from the following commutative diagram:



The arrows with hook denote induced subfans (see Example 4.8.(c) and Proposition 4.27), the ones with double head refinements (see Example 4.8.(d) and Definition 4.24). The left square is given by definition of Δ_{M} and $\widetilde{\Delta}_{\mathsf{M}}$ (see Notations 4.10 and 4.30). In particular, $\left|\widetilde{\Delta}_{\mathsf{M}}\right| = |\Delta_{\mathsf{M}}|$. The right square realizes Definition 4.24 of $\Sigma_{-\mathsf{M},\mathsf{M}^\perp}$. The map δ is the defining feature of $\Sigma_{E,E}$ (see Example 4.8.(d)). Commutativity of the upper left triangle is due to the identity $\pi_2 \circ \mu = \delta$ from Lemma 4.9. The lower triangles exist trivially because the first component of μ is the identity. □

4.5. **Resolution of configuration hypersurfaces.** In this last subsection, we examine the tropical resolution given by the square conormal fan. By composing with the normalized Nash blow-up, we arrive at an explicit resolution of the configuration hypersurface, proving Theorem 1.2 from the introduction.

Definition 4.32. For any configuration $W \subseteq V$, set $\widetilde{\Lambda}_W := \Lambda_W(\widetilde{\Delta}_{\mathsf{M}_W})$. $\diamondsuit$

Corollary 4.33. *Let $W \subseteq V$ be a connected configuration. Then $\widetilde{\Lambda}_W$ is a smooth schön compactification of $\Lambda_W^\circ \subseteq \mathbb{T}_{E,E^\star}$ and the projection $\pi_2\colon N_{E,E} \to N_E$ induces a resolution of singularities of X_W, which factors through Λ_W:*

$$\begin{array}{ccc} \widetilde{\Lambda}_W & \xrightarrow{\quad q \quad} & \Lambda_W \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X_W & \end{array} \tag{4.34}$$

Proof. Combine Theorems 4.21 and 4.31 using Corollary 4.23. □

As noted in the introduction, a key feature of a (schön) tropical compactification is its simple normal crossings boundary. For each square biflat $F \subsetneq G$, there is a ray $e_{E\setminus F} + f_{G\setminus F}$ in $\widetilde{\Delta}_W$ that indexes a torus-invariant divisor in $\mathbb{P}(\widetilde{\Delta}_W)$ which we will denote $D_W^{F\subseteq G}$. We further let $\widetilde{D}_W^{F\subseteq G} = D_W^{F\subseteq G} \cap \widetilde{\Lambda}_W$.

Proposition 4.35. *Let $W \subseteq V$ be a connected configuration. Then*

$$\widetilde{\Lambda}_W \setminus \Lambda_W^\circ = \bigcup_{F\subseteq G} \widetilde{D}_W^{F\subseteq G},$$

the union running over all all square biflats $F \subseteq G$. Each $\widetilde{D}_W^{F\subseteq G}$ is a smooth divisor, and $\bigcap_{1\le i\le k} D_{F_i\subseteq G_i} \neq \varnothing$ if and only if the bisubsets $(E\setminus F_i)|G_i$, $1 \le i \le k$, form a biflag.

Proof. By Corollary 4.33, $\tilde{\Lambda}_W$ is schön. By results of Tevelev and Hacking (see [Hac08, Thm. 2.4, Lem. 2.7]) it intersects the $\mathbb{T}$-orbits in smooth varieties of expected dimension.

Under the schön hypothesis, the multiplication map in the toric variety is smooth, which implies that the torus-invariant divisors in $\mathbb{P}(\widetilde{\Delta}_W)$ intersect $\widetilde{\Lambda}_W$ transversely. The divisor $D_W^{F\subseteq G}$ is is a toric subvariety of $\mathbb{P}(\widetilde{\Delta}_W)$, hence smooth; these facts together imply that the intersection $\widetilde{D}_W^{F\subseteq G}$ is also smooth. □

The fibres of the maps $\widetilde{\Lambda}_W \to \Lambda_W \to X_W$ are controlled by the ambient toric variety. By definition, the first map is an isomorphism over the torus, and Λ_W° is smooth by Proposition 4.4. Now we consider its boundary by looking at its intersections with coordinate subspaces.

Lemma 4.36. *Let $W \subseteq V$ be a configuration with connected matroid $\mathsf{M} = \mathsf{M}_W$. For any point $(w, \beta) \in \Lambda_W \setminus \Lambda_W^\circ$, the pair of sets $F(w) \subsetneq F(w) \cup F(\beta)$ is a square biflat.*

Proof. We may work affinely, so let $(w, \beta) \in \hat{\Lambda}_W \setminus \hat{\Lambda}_W^\circ$ with $w \neq 0$ and $\beta \neq 0$. Then $w \in W$ and $F(w)$ is a flat of M. By definition of $\hat{\Lambda}_W$, $Q_W^\star(w, \beta) = 0$. Equivalently, $w\beta := Q_E^\star(w, \beta) \in W^\perp$, where

$$(w\beta)_i = \beta_w(x_i) = \beta \circ Q_E(w, x_i) = \beta \left(\sum_{i\in E} w_i x_i\right) = \sum_{i\in E} w_i \beta_i.$$

Thus, $F(w\beta) = F(w) \cup F(\beta)$ is a flat of $\mathsf{M}^\perp$. If $F(w) = \varnothing$, then $F(w\beta) = F(\beta)$ is neither empty since $\beta \notin \hat{\mathbb{T}}_{E^\star}$, nor equal to E since $\beta \neq 0$. The conclusion follows. □

For any configuration $W \subseteq V$ and each pair $(F, S) \in \mathscr{L}_{\mathsf{M}_W} \times 2^E$ let

$$\begin{aligned}\Lambda_W^{F,S} &\coloneqq \Lambda_W \cap \big(\mathbb{P}(\mathbb{K}^{E\setminus F}) \times \mathbb{P}(\mathbb{K}^{E\setminus S})\big)\\ &= \{(w,\beta) \in \Lambda_W \mid F(w) \supseteq F,\ F(\beta) \supseteq S\}.\end{aligned}$$

Its preimage in $\widetilde{\Lambda}_W$ is an intersection with a toric variety:

Proposition 4.37. *Let $W \subseteq V$ be a configuration with connected matroid $\mathsf{M} = \mathsf{M}_W$. Then the preimage under q from (4.34) of $\Lambda_W^{F,S}$ for $(F,S) \in \mathscr{L}_{\mathsf{M}_W} \times 2^E$ is*

$$q^{-1}(\Lambda_W^{F,S}) = \widetilde{\Lambda}_W \cap \mathbb{P}(\widetilde{\Delta}_{\mathsf{M}}^{F,S}),$$

where $\widetilde{\Delta}_{\mathsf{M}}^{F,S}$ is the subfan of $\widetilde{\Delta}_{\mathsf{M}}$ induced on rays indexed by square biflats $F' \subseteq G'$, where $F' \subseteq F$ and $G' \setminus F' \subseteq S$.

Proof. By construction, $\Lambda_W^{F,S}$ is the intersection of Λ_W with the toric variety given by the star of the cone

$$\sigma_{F,S} \coloneqq \mathbb{R}_{\geq 0}\big(\{e_i \mid i \in F\} \cup \{f_j \mid j \in S\}\big). \tag{4.38}$$

Then $q^{-1}(\mathbb{P}(\mathrm{star}(\sigma_{F,S})))$ is the toric variety given by the induced subfan of $\widetilde{\Delta}_{\mathsf{M}}$ on rays that intersect relint $\sigma_{F,S}$ (see, e.g., [CLS11, Lem. 3.3.21]). Rays of $\widetilde{\Delta}_{\mathsf{M}}$ are spanned by vectors $e_{F'} + f_{G'\setminus F'}$, where $F' \subseteq G'$ is a square biflat. Such a ray intersects relint $\sigma_{F,S}$ if and only if $F' \subseteq F$ and $G' \setminus F' \subseteq S$. The conclusion follows from Proposition 4.35. □

Example 4.39 (Example 3.15, continued)**.** In this example, $\mathsf{M} = \mathsf{M}_W$ was not round, so Λ_W was not smooth. Here we describe the resolution $\widetilde{\Lambda}_W$ given by the square conormal fan. First, the (proper) flats of M and $\mathsf{M}^\perp$, respectively, equal

$$\{1, 2, 3, 4, 5, 124, 135, 23, 25, 34, 45\} \text{ and } \{1, 24, 35\},$$

so the square biflats are

$$\begin{array}{llllllll} i \subseteq E, & 124 \subseteq E, & 135 \subseteq E, & 23 \subseteq E, & 25 \subseteq E, & 34 \subseteq E, & 45 \subseteq E, & \\ 1 \subseteq 1, & 2 \subseteq 24, & 3 \subseteq 35, & 4 \subseteq 24, & 5 \subseteq 25, & \varnothing \subseteq 1, & \varnothing \subseteq 24, & \varnothing \subseteq 35 \end{array}$$

for $i = 1, \ldots, 5$. The square biflat $F \subseteq G$ gives a ray in the direction $-e_F + f_G$ in $\Sigma_{-\mathsf{M},\mathsf{M}^\perp}$, which maps to $e_F - f_{G\setminus F}$ under the the linear isomorphism $-\mu\colon \Sigma_{-\mathsf{M},\mathsf{M}^\perp} \cong \widetilde{\Delta}_{\mathsf{M}}$. There are 56 maximal cones.

Fibres of $q\colon \widetilde{\Lambda}_W \to \Lambda_W$ can be understood by restricting the map of ambient toric varieties. For example, we recall that Λ_W has two singular points, (α_{124}, β_0) and (α_{135}, β_0), which are the strata $\Lambda_W^{124,2345}$ and $\Lambda_W^{135,2345}$ respectively, while

$$\Lambda_W^{1,2345} = \big\{(w_{st}, \beta_0) \mid (s:t) \in \mathbb{P}^1\big\}$$

is a line inside Λ_W, where $w_{st} \coloneqq s\alpha_{124} + t\alpha_{135}$.

Using Proposition 4.37, we find four square biflats $\varnothing \subseteq 24$, $\varnothing \subseteq 35$, $1 \subseteq 1$ and $1 \subseteq E$ indexing the rays of the 2-dimensional fan $\Delta_{W;1,2345}$, shown in bold in Figure 2. Then $q^{-1}(\Lambda_W^{1,2345})$ is a union of four divisors in $\widetilde{\Lambda}_W$ that meet along three curves.

At $t = 0$, the fan $\Delta_{W;124,2345} \supseteq \Delta_{W;1,2345}$ is obtained by adding three more rays indexed by $2 \subseteq 24$, $4 \subseteq 24$, and $124 \subseteq E$: the last ray is in every maximal cone. This fan has dimension 3, so $q^{-1}(\{(\alpha_{124}, \beta_0)\})$ is a union of seven smooth divisors in Λ_W that meet in pairs along 11 curves and in threes at five points. For $s = 0$, the situation is the same up to symmetry. The fans for the toric varieties intersecting $q^{-1}(\Lambda_W^{1,2345})$ are shown schematically in Figure 2.

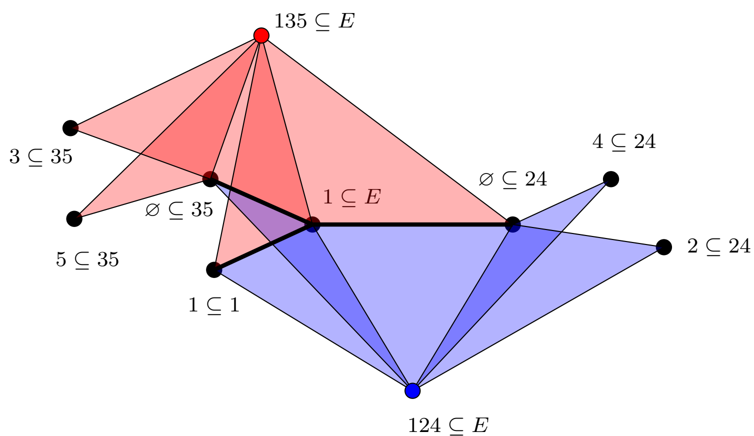


FIGURE 2. Boundary structure of $q^{-1}(\Lambda_W^{1,2345})$

$\diamondsuit$

REFERENCES

[ADH23] Federico Ardila, Graham Denham, and June Huh. "Lagrangian geometry of matroids." In: *J. Amer. Math. Soc.* 36.3 (2023), pp. 727–794.

[AHK18] Karim Adiprasito, June Huh, and Eric Katz. "Hodge theory for combinatorial geometries." In: *Ann. of Math. (2)* 188.2 (2018), pp. 381–452.

[AK06] Federico Ardila and Caroline J. Klivans. "The Bergman complex of a matroid and phylogenetic trees." In: *J. Combin. Theory Ser. B* 96.1 (2006), pp. 38–49.

[BB03] Prakash Belkale and Patrick Brosnan. "Matroids, motives, and a conjecture of Kontsevich." In: *Duke Math. J.* 116.1 (2003), pp. 147–188.

[BEK06] Spencer Bloch, Hélène Esnault, and Dirk Kreimer. "On motives associated to graph polynomials." In: *Comm. Math. Phys.* 267.1 (2006), pp. 181–225.

[Blo] Spencer Bloch. *A note on Hodge structures associated to graphs*. unpublished.

[Blo20] Spencer Bloch. "Feynman amplitudes in mathematics and physics." In: *Amplitudes, Hodge theory and ramification—from periods and motives to Feynman amplitudes*. Ed. by K. Ebrahimi-Fard, J. I. Burgos Gil, and D. Manchon. Vol. 21. Clay Mathematics Proceedings. American Mathematical Society, Providence, RI; for the Clay Mathematics Institute, Cambridge, MA, 2020, pp. xiv+229.

[BM26] Daniel Bath and Mircea Mustaţă. *On singularities of determinantal hypersurfaces*. 2026. arXiv: 2601.22072.

[BMW24] Daniel Bath, Mircea Mustaţă, and Uli Walther. *Singularities of square-free polynomials*. 2024. arXiv: 2412.11309.

[Bor16] Svetlana Borissova. *Regular round matroids*. MSc thesis. 2016.

[BW24] Daniel Bath and Uli Walther. *Matroidal polynomials, their singularities, and applications to Feynman diagrams*. 2024. arXiv: 2404.07885.

[CDM24] Qianyu Chen, Bradley Dirks, and Mircea Mustaţă. "The minimal exponent and k-rationality for local complete intersections." In: *J. Éc. polytech. Math.* 11 (2024), pp. 849–873.

[Che+24] Qianyu Chen, Bradley Dirks, Mircea Mustaţă, and Sebastián Olano. "V-filtrations and minimal exponents for local complete intersections." In: *J. Reine Angew. Math.* 811 (2024), pp. 219–256.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2011, pp. xxiv+841.

[Den+22] Graham Denham, Delphine Pol, Mathias Schulze, and Uli Walther. "Configuration polynomials under contact equivalence." In: *Ann. Inst. Henri Poincaré D* 9.4 (2022), pp. 793–812.

[DM24] Rankeya Datta and Takumi Murayama. "Permanence properties of F-injectivity." In: *Math. Res. Lett.* 31.4 (2024), pp. 985–1027.

[DSW21] Graham Denham, Mathias Schulze, and Uli Walther. "Matroid connectivity and singularities of configuration hypersurfaces." In: *Lett. Math. Phys.* 111.11 (2021), pp. 1–67.

[Fed83] Richard Fedder. "F-purity and rational singularity." In: *Trans. Amer. Math. Soc.* 278.2 (1983), pp. 461–480.

[FS05] Eva-Maria Feichtner and Bernd Sturmfels. "Matroid polytopes, nested sets and Bergman fans." In: *Port. Math. (N.S.)* 62.4 (2005), pp. 437–468.

[Ful98] William Fulton. *Intersection theory*. Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998, pp. xiv+470.

[GGW03] James F. Geelen, A. M. H. Gerards, and Geoff Whittle. "Disjoint cocircuits in matroids with large rank." In: *J. Combin. Theory Ser. B* 87.2 (2003), pp. 270–279.

[GKM09] Andreas Gathmann, Michael Kerber, and Hannah Markwig. “Tropical fans and the moduli spaces of tropical curves.” In: *Compos. Math.* 145.1 (2009), pp. 173–195.

[GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Reprint of the 1994 edition. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. x+523.

[GS] Daniel R. Grayson and Michael E. Stillman. *Macaulay2, a software system for research in commutative algebra and algebraic geometry*. Available at http://www2.macaulay2.com.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Schemes with examples and exercises. Vieweg + Teubner, Wiesbaden, 2010, pp. viii+615.

[Hac08] Paul Hacking. “The homology of tropical varieties.” In: *Collect. Math.* 59.3 (2008), pp. 263–273.

[Har98a] Nobuo Hara. “A characterization of rational singularities in terms of injectivity of Frobenius maps.” In: *Amer. J. Math.* 120.5 (1998), pp. 981–996.

[Har98b] Nobuo Hara. “A characterization of rational singularities in terms of injectivity of Frobenius maps.” In: *Amer. J. Math.* 120.5 (1998), pp. 981–996.

[Has10] Mitsuyasu Hashimoto. “F-pure homomorphisms, strong F-regularity, and F-injectivity.” In: *Comm. Algebra* 38.12 (2010), pp. 4569–4596.

[HH94] Melvin Hochster and Craig Huneke. “F-regularity, test elements, and smooth base change.” In: *Trans. Amer. Math. Soc.* 346.1 (1994), pp. 1–62.

[HMS14] Jun Horiuchi, Lance Edward Miller, and Kazuma Shimomoto. “Deformation of F-injectivity and local cohomology.” In: *Indiana Univ. Math. J.* 63.4 (2014). With an appendix by Karl Schwede and Anurag K. Singh, pp. 1139–1157.

[HU87] Craig Huneke and Bernd Ulrich. “The structure of linkage.” In: *Ann. of Math. (2)* 126.2 (1987), pp. 277–334.

[HW02] Nobuo Hara and Kei-Ichi Watanabe. “F-regular and F-pure rings vs. log terminal and log canonical singularities.” In: *J. Algebraic Geom.* 11.2 (2002), pp. 363–392.

[Kem86] George R. Kempf. “Varieties with rational singularities.” In: *The Lefschetz centennial conference, Part I (Mexico City, 1984)*. Vol. 58. Contemp. Math. Amer. Math. Soc., Providence, RI, 1986, pp. 179–182.

[Kol95] János Kollár. *Shafarevich maps and automorphic forms*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1995, pp. x+201.

[Kun86] Joseph P. S. Kung. “Numerically regular hereditary classes of combinatorial geometries.” In: *Geom. Dedicata* 21.1 (1986), pp. 85–105.

[LQ11] Mark Luxton and Zhenhua Qu. “Some results on tropical compactifications.” In: *Trans. Amer. Math. Soc.* 363.9 (2011), pp. 4853–4876.

[MO15] David Mumford and Tadao Oda. *Algebraic geometry. II.* Vol. 73. Texts and Readings in Mathematics. Hindustan Book Agency, New Delhi, 2015, pp. x+504.

[MP22a] Linquan Ma and Thomas Polstra. *F-singularities: a commutative algebra approach.* https://www.math.purdue.edu/ ma326/F-singularitiesBook.pdf. 2022.

[MP22b] Mircea Mustaţă and Mihnea Popa. "Hodge filtration on local cohomology, Du Bois complex and local cohomological dimension." In: *Forum Math. Pi* 10 (2022), Paper No. e22, 58.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry.* Vol. 161. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015, pp. xii+363.

[Oxl11] James Oxley. *Matroid theory.* Second Edition. Vol. 21. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2011, pp. xiv+684.

[Pat10] Eric Patterson. "On the singular structure of graph hypersurfaces." In: *Commun. Number Theory Phys.* 4.4 (2010), pp. 659–708.

[PS74] C. Peskine and L. Szpiro. "Liaison des variétés algébriques. I." In: *Invent. Math.* 26 (1974), pp. 271–302.

[PT24] Vaibhav Pandey and Yevgeniya Tarasova. "Linkage and F-regularity of determinantal rings." In: *Int. Math. Res. Not. IMRN* 11 (2024), pp. 9323–9339.

[Sai93] Morihiko Saito. "On b-function, spectrum and rational singularity." In: *Math. Ann.* 295.1 (1993), pp. 51–74.

[Sch09] Karl Schwede. "F-injective singularities are Du Bois." In: *Amer. J. Math.* 131.2 (2009), pp. 445–473.

[Smi97] Karen E. Smith. "F-rational rings have rational singularities." In: *Amer. J. Math.* 119.1 (1997), pp. 159–180.

[Sta23] The Stacks project authors. *The Stacks project.* https://stacks.math.columbia.edu. 2023.

[Tev07] Jenia Tevelev. "Compactifications of subvarieties of tori." In: *Amer. J. Math.* 129.4 (2007), pp. 1087–1104.

[TW18] Shunsuke Takagi and Kei-Ichi Watanabe. "F-singularities: applications of characteristic p methods to singularity theory [translation of MR3135334]." In: *Sugaku Expositions* 31.1 (2018), pp. 1–42.

[Vél95] Juan D. Vélez. "Openness of the F-rational locus and smooth base change." In: *J. Algebra* 172.2 (1995), pp. 425–453.

[Wie69] Hartmut Wiebe. "Über homologische Invarianten lokaler Ringe." In: *Math. Ann.* 179 (1969), pp. 257–274.

Daniel Bath, Departement Wiskunde, KU Leuven, 3001 Leuven, Belgium
Email address: dan.bath@kuleuven.be

Graham Denham, Department of Mathematics, University of Western Ontario, London, ON, Canada N6A 5B7
Email address: gdenham@uwo.ca

Mathias Schulze, Department of Mathematics, RPTU University Kaiserslautern-Landau, 67663 Kaiserslautern, Germany
Email address: mschulze@rptu.de

U. Walther, Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
Email address: walther@purdue.edu