Complete sets of logarithmic vector fields
for integration-by-parts identities of Feynman integrals

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Integration-by-parts identities between loop integrals arise from the vanishing integration of total derivatives in dimensional regularization. Generic choices of total derivatives in the Baikov or parametric representations lead to identities which involve dimension shifts. These dimension shifts can be avoided by imposing a certain constraint on the total derivatives. The solutions of this constraint turn out to be a specific type of syzygies which correspond to logarithmic vector fields along the Gram determinant formed of the independent external and loop momenta. We present an explicit generating set of solutions in Baikov representation, valid for any number of loops and external momenta, obtained from the Laplace expansion of the Gram determinant. We provide a rigorous mathematical proof that this set of solutions is complete. This proof relates the logarithmic vector fields in question to ideals of submaximal minors of the Gram matrix and makes use of classical resolutions of such ideals.

I. INTRODUCTION

The increasing precision of the experimental measurements of scattering processes at the Large Hadron Collider (LHC) is calling for increased precision in the theoretical prediction of cross sections. The computations of the leading-order (LO) and next-to-leading-order (NLO) contributions to the cross sections are by now automated, but for many processes the next-to-next-to-leading-order (NNLO) contribution is needed to reach the required precision.

The NNLO cross section has double-real, real-virtual and double-virtual contributions. The aim of this paper is to provide new tools for computing the latter contributions, i.e., the two-loop scattering amplitudes. Results for the latter may in turn motivate progress on the combination of all virtual and real contributions to the NNLO cross section.

A key tool in the calculation of multi-loop amplitudes are integration-by-parts (IBP) reductions. The latter arise from the vanishing integration of total derivatives in dimensional regularization,

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu P}{D_1^{\nu_1} \cdots D_m^{\nu_m}} = 0, \qquad (1.1)$$

where P and the vectors v_j^μ are polynomial in the loop momenta ℓ_i and external momenta, the D_k denote inverse propagators, and the ν_i are integers. The IBP identities turn out to generate a large set of linear relations between loop integrals. This then allows for most integrals to be reexpressed as a linear combination of basis integrals. In practice, the basis contains much fewer integrals than the number of integrals produced by the Feynman rules for a multi-loop amplitude.

The step of performing Gaussian elimination on the linear systems that arise from eq. (1.1) may be carried out with the Laporta algorithm [1, 2], which leads in general to relations that involve integrals with squared propagators. There are various publicly available implementations of automated IBP reduction: AIR [3], FIRE [4, 5], Reduze [6, 7], LiteRed [8], Kira [9], as well as private implementations. A formalism for obtaining IBP reductions without squared propagators was developed in refs. [10, 11]. A systematic method of deriving IBP reductions on generalized-unitarity cuts was given in ref. [12]. A recent approach [13] uses sampling over finite fields to construct the reduction coefficients. Other recent developments include software for determining a basis of integrals [14] and a D-module theory based approach for computing the number of basis integrals [15].

The IBP reductions moreover allow setting up differential equations for the basis integrals, thereby enabling their evaluation. [16–24]. Differential equations have proven to be a powerful tool for calculating multi-loop integrals, enabling for example the computation of the basis integrals for numerous two-loop amplitudes of $2 \to 2$ processes. This method can therefore reasonably be expected to be of relevance to two-loop amplitudes

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for $2 \to 3$ processes. We note that, in the context of the latter, impressive results have recently appeared [25–28].

In this paper we study integration-by-parts identities (1.1). Generic choices of total derivatives in the Baikov or parametric representations lead to identities which involve undesirable dimension shifts. These dimension shifts can be avoided by imposing a certain constraint on the total derivatives. The solutions of this constraint turn out to be a specific type of syzygies which correspond to logarithmic vector fields along the Gram determinant formed of the independent external and loop momenta. We will present an explicit generating set of solutions in Baikov representation, valid for any number of loops and external momenta, obtained from the Laplace expansion of the Gram determinant. We will then present a rigorous mathematical proof that this set of solutions is complete. This proof relates the logarithmic vector fields in question to ideals of submaximal minors of the Gram matrix and makes use of classical resolutions of such ideals.

An important feature of the obtained generating set of syzygies is that they are guaranteed to have degree at most one. In contrast, a generating set of syzygies obtained from an S-polynomial computation would in general have higher degrees. The fact that the syzygies obtained here are of degree at most one is useful because it dramatically simplifies the computation of solutions which satisfy further constraints. For example, one may be interested in imposing the further constraint on the total derivatives that no integrals with squared propagators are encountered in the integration-by-parts identities.

This paper is organized as follows. In Sec. II we set up notation and give the Baikov representation of a generic Feynman loop integral. In Sec. III we study integration-by-parts relations on unitarity cuts and derive the syzygy equation of interest. In Sec. IV we obtain a closed-form generating set of solutions to the syzygy equation. In Sec. V we present a proof that the set of syzygies is complete. In Sec. VI we provide an example of the formalism. In Sec. VII we give our conclusions.

II. BAIKOV REPRESENTATION OF LOOP INTEGRALS

In this paper we will make use of the Baikov representation of Feynman loop integrals. As will be explained later, this parametrization is useful for our purpose of studying integration-by-parts identities (1.1) on so-called cuts where some number of propagators are put on shell, i.e. after evaluating the residue at $D_\alpha = 0$. Since the Baikov representation uses the inverse propagators D_α as variables, it greatly facilitates the application of cuts.

In this section we fix our notations and review the Baikov representation of a general Feynman loop integral. We consider an integral with L loops and k propagators. We denote the loop momenta as $\ell_1, \ldots, \ell_L$ and the external momenta as $p_1, \ldots, p_E, p_{E+1}$, where E thus denotes the number of linearly independent external mo-

menta. Furthermore, the integrand may involve $m - k$ irreducible scalar products—that is, polynomials in the loop momenta and external momenta which cannot be expressed as a linear combination of the inverse propagators. As will be shown below, m is a function of L and E. We apply dimensional regularization to regulate infrared and ultraviolet divergences and normalize the integral as follows,

$$I(\nu_1, \ldots, \nu_m; D) \equiv \int \prod_{j=1}^{L} \frac{\mathrm{d}^D \ell_j}{\mathrm{i}\pi^{D/2}} \frac{N_{k,m}}{D_1^{\nu_1} \cdots D_k^{\nu_k}}, \quad \nu_i \geq 0. \tag{2.1}$$

The inverse propagators D_j are of the form P^2 where P is an integer-coefficient linear combination of vectors taken from the ordered set of all independent external and loop momenta,

$$V = (v_1, \ldots, v_{E+L}) = (p_1, \ldots, p_E, \ell_1, \ldots, \ell_L). \tag{2.2}$$

Furthermore, the quantity $N_{k,m}$ in eq. (2.1) is defined as $N_{k,m} \equiv D_{k+1}^{\nu_{k+1}} \cdots D_m^{\nu_m}$.

We now proceed to present the Baikov representation [29] of the integral (2.1). To this end, we start by writing down the Gram matrix S of the independent external and loop momenta,

$$S = \left(\begin{array}{ccc|ccc} x_{1,1} & \cdots & x_{1,E} & x_{1,E+1} & \cdots & x_{1,E+L} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{E,1} & \cdots & x_{E,E} & x_{E,E+1} & \cdots & x_{E,E+L} \\ \hline x_{E+1,1} & \cdots x_{E+1,E} & x_{E+1,E+1} & \cdots x_{E+1,E+L} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{E+L,1} & \cdots x_{E+L,E} & x_{E+L,E+1} & \cdots x_{E+L,E+L} \end{array} \right), \tag{2.3}$$

where the entries are given by,

$$x_{i,j} = v_i \cdot v_j, \tag{2.4}$$

where v_i and v_j are entries of V in eq. (2.2). In addition, we let F denote the determinant of S,

$$F \equiv \det S. \tag{2.5}$$

The entries of the upper-left $E \times E$ block of S are constructed out of the external momenta only, and it will be convenient for the following to emphasize this by relabeling these entries,

$$\lambda_{i,j} = x_{i,j} \quad \text{for} \quad 1 \leq i, j \leq E. \tag{2.6}$$

Furthermore we define G as the Gram matrix of the independent external momenta,

$$G = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,E} \\ \vdots & \ddots & \vdots \\ \lambda_{E,1} & \cdots & \lambda_{E,E} \end{pmatrix}, \tag{2.7}$$

and let U denote its determinant,

$$U = \det G \,. \qquad (2.8)$$

We remark that U is equal to the square of the volume of the parallelotope formed by the independent external momenta $\{p_1, \ldots, p_E\}$. Thus, U is non-vanishing provided that $p_1, \ldots, p_E$ are not linearly dependent.

The entries of the remaining blocks of S depend on the loop momenta. As S is a symmetric matrix, not all entries are independent. We can choose as a set of independent entries for example the entries of the upper-right $E \times L$ block along with the upper-triangular entries of the lower-right $L \times L$ block,

$$x_{i,j} \quad \text{where} \quad \begin{cases} 1 \le i \le E \quad \text{and} \quad E{+}1 \le j \le E{+}L \,, \\ E{+}1 \le i \le j \le E{+}L \,. \end{cases} \qquad (2.9)$$

Hence we find that S contains $LE + \frac{L(L+1)}{2}$ independent entries which depend on the loop momenta. From the fact that each inverse propagator D_α is the square of a linear combination of the elements of V in eq. (2.2) and the fact that the elements of V are linearly independent, it follows that D_α can be written as a unique linear combination of the $x_{i,j}$ in eq. (2.9). We therefore conclude that the combined number of propagators and irreducible scalar products in eq. (2.1) is given by the expression,

$$m = LE + \frac{L(L+1)}{2} \,. \qquad (2.10)$$

Keeping the relabeling in eq. (2.6) in mind, we can write any inverse propagator D_α (with $\alpha = 1, \ldots, m$) as an explicit linear combination of the $x_{i,j}$ in eq. (2.9) as follows,

$$D_\alpha = \sum_{\beta=1}^{m} A_{\alpha,\beta} x_\beta + \sum_{1 \le k \le l \le E} (B_\alpha)_{k,l} \lambda_{k,l} - m_\alpha^2 \,, \quad (2.11)$$

where $A_{\alpha,\beta}$ and the entries of B_α are integers. In writing this expression we introduced a lexicographic order on the set of elements (i,j) in eq. (2.9) and let $\beta = 1, \ldots, m$ denote the element label in the ordered set.

The variables of the Baikov representation [29] are chosen as the inverse propagators and the irreducible scalar products,

$$z_\alpha \equiv D_\alpha \quad \text{where} \quad 1 \le \alpha \le m \,. \qquad (2.12)$$

We can now present the Baikov representation of the integral in eq. (2.1). It takes the following form[1],

$$I(\nu; D) = C_E^L \, U^{\frac{E-D+1}{2}} \int \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_m}{z_1^{\nu_1} \cdots z_k^{\nu_k}} F^{\frac{D-L-E-1}{2}} N_{k,m} \,, \qquad (2.13)$$

where the first prefactor is given by the expression,

$$C_E^L \equiv \frac{\pi^{-L(L-1)/4 - LE/2}}{\prod_{j=1}^{L} \Gamma\left(\frac{D-L-E+j}{2}\right)} \det A \,, \qquad (2.14)$$

where A is the matrix defined in eq. (2.11).

III. INTEGRATION-BY-PARTS IDENTITIES ON UNITARITY CUTS

In this section we consider integration-by-parts identities (1.1) on cuts where some number of propagators are put on shell, i.e. roughly speaking $\frac{1}{D_i} \to \delta(D_i)$. This has the advantage of reducing the linear systems to which Gauss-Jordan elimination is to be applied. As explained in ref. [12], it is possible to determine complete integration-by-parts reductions by performing the reductions on a suitably chosen spanning set of cuts and merge the information found on each cut.

The virtue of the Baikov representation (2.13) is that it makes manifest the effect of cutting propagators. Cf. refs. [11, 12], we consider applying a c-fold cut (where $0 \le c \le k$) to eq. (2.13). We let $\mathcal{S}_{\text{cut}}$, $\mathcal{S}_{\text{uncut}}$ and $\mathcal{S}_{\text{ISP}}$ denote the sets of indices labeling cut propagators, uncut propagators and irreducible scalar products respectively, and set,

$$\begin{aligned} \mathcal{S}_{\text{cut}} &= \{\zeta_1, \ldots, \zeta_c\} \\ \mathcal{S}_{\text{uncut}} &= \{r_1, \ldots, r_{k-c}\} \\ \mathcal{S}_{\text{ISP}} &= \{r_{k-c+1}, \ldots, r_{m-c}\} \,. \end{aligned} \qquad (3.1)$$

We will restrict the analysis to the case where the propagator powers in eq. (2.1) are equal to one, $\nu_i = 1$.

The result of applying the cut,

$$\int \frac{\mathrm{d}z_i}{z_i} \quad \xrightarrow{\text{cut}} \quad \oint_{\Gamma_\epsilon(0)} \frac{\mathrm{d}z_i}{z_i} \quad \text{where} \quad i \in \mathcal{S}_{\text{cut}} \,, \quad (3.2)$$

where $\Gamma_\epsilon(0)$ denotes a circle centered at 0 of radius $\epsilon > 0$ to eq. (2.13) is obtained by evaluating the residue at $z_i = 0$ where $i \in \mathcal{S}_{\text{cut}}$,

$$\begin{aligned} I_{\text{cut}}(\nu; D) = {}& C_E^L(D) U^{\frac{E-D+1}{2}} \\ &\times \int \frac{\mathrm{d}z_{r_1} \cdots \mathrm{d}z_{r_{m-c}} N_{k,m}}{z_{r_1} \cdots z_{r_{k-c}}} F(z)^{\frac{D-L-E-1}{2}} \bigg|_{z_i = 0, \; i \in \mathcal{S}_{\text{cut}}} \,. \end{aligned} \qquad (3.3)$$

We now turn to integration-by-parts identities evaluated on the c-fold cut $\mathcal{S}_{\text{cut}}$. Such identities correspond

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[1] We remark that the Baikov representation in eq. (2.13) is consistent with that used in ref. [12]. This is a consequence of the identity $\det_{i,j=1,\ldots,L} \mu_{i,j} = \frac{F}{U}$, which in turn follows from the Schur complement theorem in linear algebra. Moreover, ref. [12] makes use of the four-dimensional helicity scheme. It is there-

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fore imposed as a constraint on the external momenta that they span a vector space of dimension at most four. In order words, one must have $\dim \text{span}\{p_1, \ldots, p_E\} \le 4$. Accordingly, the exponent of the Baikov polynomial F is modified from $\frac{D-L-E-1}{2}$ in eq. (2.13) to $\frac{D-L-5}{2}$ and $\frac{D-L-4}{2}$ for $E \ge 4$ and $E = 3$, respectively.

to exact differential forms of degree $m - c$. The most general exact differential form which is of the form of the integrand of eq. (3.3) is,

$$0 = \int \mathrm{d}\Big(\sum_{i=1}^{m-c} \frac{(-1)^{i+1} a_{r_i} F(z)^{\frac{D-L-E-1}{2}}}{z_{r_1} \cdots z_{r_{k-c}}}$$
$$\times \mathrm{d}z_{r_1} \wedge \cdots \wedge \widehat{\mathrm{d}z_{r_i}} \wedge \cdots \wedge \mathrm{d}z_{r_{m-c}} \Big), \quad (3.4)$$

where the a_i are polynomials in $\{z_{r_1}, \ldots, z_{r_{m-c}}\}$. Expanding eq. (3.4), we get an integration-by-parts identity,

$$0 = \int \Big[\sum_{i=1}^{m-c} \Big(\frac{\partial a_{r_i}}{\partial z_{r_i}} + \frac{D-L-E-1}{2F(z)} a_{r_i} \frac{\partial F}{\partial z_{r_i}} \Big) - \sum_{i=1}^{k-c} \frac{a_{r_i}}{z_{r_i}} \Big]$$
$$\times \frac{F(z)^{\frac{D-L-E-1}{2}}}{z_{r_1} \cdots z_{r_{k-c}}} \mathrm{d}z_{r_1} \cdots \mathrm{d}z_{r_{m-c}}. \quad (3.5)$$

We observe that, for an arbitrary choice of polynomials $a_i(z)$, the two terms in the parenthesis $(\cdots)$ in eq. (3.5) correspond to integrals in D and $D-2$ dimensions, respectively. This is because the $\frac{1}{F(z)}$ factor in the second term has the effect of modifying the integration measure, thereby shifting the space-time dimension from D to $D-2$.

To get the exact form in eq. (3.4) to correspond to an integration-by-parts relation in D dimensions, we require the $a_i(z)$ to be chosen such that,

$$bF + \sum_{i=1}^{m-c} a_{r_i} \frac{\partial F}{\partial z_{r_i}} = 0, \quad (3.6)$$

where b denotes a polynomial, since then the $\frac{1}{F}$ factor in eq. (3.5) cancels out, and no dimension shift occurs. Equations of the type (3.6) are known in algebraic geometry as *syzygy equations* (describing in our setting the polynomial relations—that is, syzygies, between $F, \frac{\partial F}{\partial z_{r_1}}, \ldots, \frac{\partial F}{\partial z_{r_{m-c}}}$). They have also been considered in the context of integration-by-parts relations in refs. [10–12, 30–32]. We remark that it follows from Schreyer's theorem that a generating set of solutions of eq. (3.6) can be found algebraically by determining a Gröbner basis of the ideal generated by the above polynomials, considering the S-polynomials involved in the Buchberger test, and expressing the corresponding relations in terms of the original generators [33]. We refer to refs. [11, 14] for a geometric interpretation of eq. (3.6).

IV. SYZYGY GENERATORS FROM LAPLACE EXPANSION

In this section we turn to obtaining a generating set $\mathcal{T} = \langle g_1, \ldots, g_d \rangle$ of syzygies $g_i = (a_{r_1}, \ldots, a_{r_{m-c}}, b)$ of eq. (3.6). By this we mean that $\mathcal{T}$ is such that any solution of eq. (3.6) can be written in the form $g_i p$ where $g_i \in \mathcal{T}$ and p denotes a polynomial.

For a general polynomial F, determining a generating set of syzygies would require an S-polynomial computation. However, as we will shortly see, in the case where F is the determinant of a matrix, a generating set of syzygies can be obtained from the Laplace expansion of the determinant of F. We remark that related work has appeared in ref. [32].

A. Off-shell case

For simplicity we start with the case where no cuts are applied, $c = 0$. Let $M = (m_{i,j})_{i,j=1,\ldots,n}$ be a generic matrix, i.e. such that all entries are independent. We consider the determinant of M and perform Laplace expansion of the determinant along the ith row,

$$\Big[\sum_{k=1}^{n} m_{j,k} \frac{\partial(\det M)}{\partial m_{i,k}} \Big] - \delta_{i,j} \det M = 0, \quad 1 \le i, j \le n. \quad (4.1)$$

The identities with $i \ne j$ follow by replacing the ith row of M by the jth row, $m_{i,k} \to m_{j,k}$, as the resulting matrix clearly has a vanishing determinant.

For a symmetric matrix $S = (s_{i,j})_{i,j=1,\ldots,n}$, the entries satisfy $s_{i,j} = s_{j,i}$ and are thus not all independent. For this case, one obtains from the Laplace expansion the following identities,

$$\Big[\sum_{k=1}^{n} (1+\delta_{i,k}) s_{j,k} \frac{\partial(\det S)}{\partial s_{i,k}} \Big] - 2\delta_{i,j} \det S = 0, \quad 1 \le i, j \le n. \quad (4.2)$$

In taking the derivatives one must take into account that the entries are not independent. To do so, we replace $s_{j,i} \to s_{i,j}$ with $i \le j$ in S before taking derivatives and furthermore replace $\frac{\partial(\det S)}{\partial s_{i,k}}$ with $i > k$ by $\frac{\partial(\det S)}{\partial s_{k,i}}$.

We will now apply the identity (4.2) to the Gram matrix S in eq. (2.3). However, before doing so, we note that the first E rows only contain external invariants $\lambda_{i,j}$ and entries which also appear in the last L rows by symmetry of S. Derivatives with respect to the $\lambda_{i,j}$ are not of interest in the problem at hand, since for integration-by-parts identities (1.1), only derivatives with respect to the loop momenta play a role. We therefore apply the identity (4.2) only to the last L rows of S, from which we find,

$$\Big[\sum_{k=1}^{E+L} (1+\delta_{i,k}) x_{j,k} \frac{\partial F}{\partial x_{i,k}} \Big] - 2\delta_{i,j} F = 0, \quad (4.3)$$

where $E+1 \le i \le E+L$ and $1 \le j \le E+L$. We can express the derivatives with respect to $x_{i,k}$ in terms of derivatives with respect to z_α by making use of the chain rule,

$$\frac{\partial F}{\partial x_{i,k}} = \sum_{\alpha=1}^{m} \frac{\partial z_\alpha}{\partial x_{i,k}} \frac{\partial F}{\partial z_\alpha} \quad \text{for} \begin{cases} 1 \le i \le E, \ E+1 \le k \le E+L, \\ E+1 \le i \le k \le E+L. \end{cases} \quad (4.4)$$

By splitting the sum in eq. (4.3) into sums over the first E, subsequent $i - 1 - E$ and $E + L - i + 1$ terms and using that $x_{i,k} = x_{k,i}$ for the former two, application of the chain rule (4.4) yields,

$$\sum_{\alpha=1}^{m} (a_{i,j})_\alpha \frac{\partial F}{\partial z_\alpha} + b_{i,j} F = 0 \,, \qquad (4.5)$$

where $a_{i,j}$ and b are given by the following expressions,

$$(a_{i,j})_\alpha = \sum_{k=1}^{E+L} (1 + \delta_{i,k}) \frac{\partial z_\alpha}{\partial x_{i,k}} x_{j,k} \quad \text{and} \quad b_{i,j} = -2\delta_{i,j} \,, \qquad (4.6)$$

for $E + 1 \leq i \leq E + L$, $1 \leq j \leq E + L$ and $1 \leq \alpha \leq m$. We conclude that

$$t_{i,j} = \Big((a_{i,j})_1, \ldots, (a_{i,j})_m, b_{i,j} \Big) \,, \qquad (4.7)$$

with $a_{i,j}$ and $b_{i,j}$ given in eq. (4.6) are solutions of eq. (3.6) in the case $c = 0$.

We note that it follows from the relations in eqs. (2.11)–(2.12) that the derivatives $\frac{\partial z_\alpha}{\partial x_{i,k}}$ are integers. Furthermore, we may use the relations to express the x-variables as a linear combination of the z-variables. This shows that the syzygies $t_{i,j}$ in eq. (4.7) are at most linear polynomials in the Baikov variables z_α.

We emphasize that the closed-form expressions in eqs. (4.6)–(4.7) are valid for any number of loops and external legs. The only quantities that depend on the graph in question are the relations of the z-variables to the x-variables in eqs. (2.11)–(2.12). We note that the approach of using Laplacian expansion to obtain syzygies works equally well in cases where the propagators are massive, since the variables $x_{i,j}$ in eq. (2.4) will be independent of the internal mass parameters. These mass parameters will appear explicitly after the linear transformation from the $x_{i,j}$ variables to the Baikov variables z_i. For an explicit example we refer to Sec. VI B.

We emphasize that the closed-form expressions allow the construction of purely D-dimensional integration-by-parts identities in cases where S-polynomial based computations of syzygies are not feasible. Another important aspect of the syzygies in eqs. (4.6)–(4.7) is that they are of degree one. This would not be guaranteed for the output of an S-polynomial-based computation of the syzygy generators which in relevant examples (see below) turn out to have higher degrees. Low-degree syzygies are particularly advantageous if we are interested in imposing additional constraints on the Ansatz for the exact form in eq. (3.5). For example, we may demand that no integrals with squared propagators are encountered in the IBP identities,

$$a_i + b_i z_i = 0 \quad \text{where} \quad i = 1, \ldots, k \,. \qquad (4.8)$$

Namely, we can obtain solutions of eqs. (3.6) and (4.8) by taking the module intersection of the module of the syzygies in eqs. (4.6)–(4.7),

$$\mathcal{T} = \big\langle t_{i,j} \;\big|\; E+1 \leq i \leq E+L \quad \text{and} \quad 1 \leq j \leq E+L \big\rangle \,, \qquad (4.9)$$

and the module,

$$\mathcal{L} = \langle z_1 e_1, \ldots, z_k e_k, e_{k+1}, \ldots, e_m \rangle \,. \qquad (4.10)$$

That is, the generators of $\mathcal{T} \cap \mathcal{L}$ form a generating set of solutions of eqs. (3.6) and (4.8) [34]. The fact that the syzygies in eqs. (4.6)–(4.7) are of degree one dramatically simplifies the computation of the module intersection $\mathcal{T} \cap \mathcal{L}$. We remark that efficient methods for computing module intersections are presented in ref. [35], and that in this reference non-trivial computations are carried out using these methods for non-planar multi-scale diagrams.

In Sec. V we give a proof that the $L(L + E)$ syzygies in eq. (4.7) form a generating set.

B. On-shell case

We now turn to obtaining a generating set of syzygies of eq. (3.6) for a generic cut $\mathcal{S}_{\text{cut}} = \{\zeta_1, \ldots, \zeta_c\}$. We start by taking the module of the syzygies in eq. (4.9) and evaluating this on the cut $\mathcal{S}_{\text{cut}}$,

$$\widehat{\mathcal{T}} = \mathcal{T}\big|_{z_q=0, \; q \in \mathcal{S}_{\text{cut}}} \,. \qquad (4.11)$$

Now, the generators $\widehat{t}_{i,j}$ of $\widehat{\mathcal{T}}$ will not in general be solutions of eq. (3.6) because the ζ_n-entries of $\widehat{t}_{i,j}$ may be nonzero on the cut.

This leads us to consider the module,

$$\mathcal{Z} = \langle e_{r_1}, \ldots, e_{r_{m-c}} \rangle \,, \qquad (4.12)$$

where e_{r_i} is an $(m+1)$-dimensional unit vector with 1 in the r_i entry and 0 elsewhere. Namely, the generators of the intersection $\widehat{\mathcal{T}} \cap \mathcal{Z}$ are solutions of eq. (3.6).

The module intersection can be found with SINGULAR and in practice takes less than a second to compute.

V. PROOF OF COMPLETENESS OF SYZYGIES

In this section we show that the $L(L + E)$ syzygies in eqs. (4.6)–(4.7) form a generating set of syzygies of eq. (3.6). In order to give a formal proof of this fact we adopt a more general setup considering polynomial logarithmic vector fields along determinants of generic (symmetric) square matrices. We reduce the problem to known resolutions of ideals of submaximal minors of such matrices.

Fix a field $\mathbb{K}$. For $0 \neq m \in \mathbb{N}$ denote by $Y = \mathbb{K}^m$ affine m-space. The coordinate ring of Y is a polynomial ring

$$\mathscr{O} = \mathscr{O}_Y = \mathbb{K}[y_1, \ldots, y_m] \,. \qquad (5.1)$$

Note that its group of units is $\mathscr{O}^* = \mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Since $\mathscr{O}$ is a Cohen–Macaulay ring, the grade or depth of any ideal of $\mathscr{O}$ equals its height or codimension (cf. Cor. 2.1.4 and Thm. 2.1.9 of ref. [36]).

The polynomial vector fields on Y form a free $\mathscr{O}$-module (cf. Prop. 16.1 of ref. [37])

$$\Theta = \Theta_Y = \mathrm{Der}_{\mathbb{K}}(\mathscr{O}) = \bigoplus_{i=1}^m \mathscr{O} \frac{\partial}{\partial y_i}\,.$$

A polynomial function

$$f\colon Y = \mathbb{K}^m \to \mathbb{K}$$

is given by an element $f \in \mathscr{O}$. The $\mathscr{O}$-submodule of Θ of logarithmic vector fields along the divisor (f) is defined by (classically for squarefree $f \in \mathscr{O} \setminus \mathscr{O}^*$ and $\mathbb{K} = \mathbb{C}$, cf. Sec. 1 of ref. [38])

$$\mathrm{Der}(-\log(f)) = \{\delta \in \Theta \mid \delta(f) \in \mathscr{O}f\} \subset \Theta_Y\,. \quad (5.2)$$

We denote the ideal of partial derivatives of f by

$$\mathscr{J}_f = \left\langle \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_m} \right\rangle\,.$$

Then $\mathrm{Der}(-\log(f))$ identifies with the projection to the first m components of the syzygy module (cf. eq. (3.6))

$$\mathrm{syz}\left(\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_m}, f \right) \cong \mathrm{syz}(\mathscr{J}_f + \mathscr{O}f)\,.$$

We call $\chi \in \mathrm{Der}(-\log(f))$ an *Euler vector field* for f if

$$\chi(f) \in \mathscr{O}^* f = \mathbb{K}^* f\,.$$

If f admits an Euler vector field, then

$$\mathrm{Der}(-\log(f)) = \mathscr{O}\chi + \mathrm{Ann}_\Theta(f)\,, \quad (5.3)$$

where

$$\mathrm{Ann}_\Theta(f) = \{\delta \in \Theta \mid \delta(f) = 0\} \cong \mathrm{syz}(\mathscr{J}_f) \quad (5.4)$$

is the annihilator of f in Θ and can be identified with the syzygy module of $\mathscr{J}_f$.

In the remainder of the section we specialize to the case where f is the determinant of a (symmetric) $n \times n$ matrix where $0 \neq n \in \mathbb{N}$. We write $\mathrm{Mat}_n(\mathscr{O})$ for the $\mathscr{O}$-module of $n \times n$ matrices with entries $\mathscr{O}$, $\mathrm{Sym}_n(\mathscr{O})$ (and $\mathrm{Skw}_n(\mathscr{O})$) for its submodule of (skew-)symmetric matrices and

$$\mathrm{Sl}_n(\mathscr{O}) = \ker(\mathrm{tr}\colon \mathrm{Mat}_n(\mathscr{O}) \to \mathscr{O})$$

the kernel of the trace map. For any $A = (a_{i,j}) \in \mathrm{Mat}_n(\mathscr{O})$ denote by $A^\star = (a_{i,j}^\star) \in \mathrm{Mat}_n(\mathscr{O})$ its adjoint matrix and by

$$I_{n-1}(A) = \left\langle a_{i,j}^\star \,\middle|\, 1 \leq i \leq j \leq n \right\rangle \lhd \mathscr{O}$$

its ideal of submaximal minors.

Consider the determinant functions

$$\det\colon X = \mathrm{Mat}_n(\mathbb{K}) \to \mathbb{K}\,, \quad \det'\colon X' = \mathrm{Sym}_n(\mathbb{K}) \to \mathbb{K}\,.$$

The preceding discussion applies to both these cases. The coordinate rings of X and X' are polynomial rings

$$\mathscr{O}_X = \mathbb{K}[x_{i,j} \mid 1 \leq i, j \leq n]\,,$$
$$\mathscr{O}_{X'} = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq j \leq n] = \mathscr{O}_X / \langle x_{i,j} - x_{j,i} \rangle\,.$$

The modules of polynomial vector fields on X and X' respectively are

$$\Theta_X = \mathrm{Der}_{\mathbb{K}}(\mathscr{O}_X) = \bigoplus_{i,j} \mathscr{O}_X \frac{\partial}{\partial x_{i,j}}\,,$$

$$\Theta_{X'} = \mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{X'}) = \bigoplus_{i \leq j} \mathscr{O}_X \frac{\partial}{\partial x_{i,j}}\,.$$

The following result provides generators of the modules of logarithmic vector fields along $f = \det$ and $f = \det'$ (cf. eq. (5.2)).

Denote by $M = (x_{i,j}) \in \mathrm{Mat}_n(\mathscr{O}_X)$ the *generic* $n \times n$ matrix and by $S = (x_{i,j}) \in \mathrm{Sym}_n(\mathscr{O}_{X'})$ its symmetric counterpart. Note that

$$\det = \det M\,, \quad \det' = \det S\,.$$

Assume from now on that $\mathbb{K}$ has characteristic different from 2 (which will be the case in our applications). Goryunov and Mond made the following observation (cf. Secs. 3.1-3.2 of ref. [39]).

Proposition 1. *There are surjective maps*

$$\mathrm{Mat}_n(\mathscr{O}_X)^{\oplus 2} \xrightarrow{\;\;\pi\;\;} \!\!\!\!\twoheadrightarrow \mathrm{Der}(-\log(\det))\,,$$
$$(A, B) \longmapsto \sum_{i,j} (MA - BM)_{i,j} \frac{\partial}{\partial x_{i,j}}\,,$$
$$\mathrm{Mat}_n(\mathscr{O}_{X'}) \xrightarrow{\;\;\pi'\;\;} \!\!\!\!\twoheadrightarrow \mathrm{Der}(-\log(\det'))\,,$$
$$A \longmapsto \sum_{i \leq j} (SA + A^t S)_{i,j} \frac{\partial}{\partial x_{i,j}}\,.$$

Proof. Since

$$\frac{\partial \det}{\partial x_{i,j}} = m_{j,i}^\star\,, \quad \frac{\partial \det'}{\partial x_{i,j}} = (2 - \delta_{i,j}) s_{i,j}^\star\,, \quad (5.5)$$

we have

$$\mathscr{J}_{\det} = I_{n-1}(M)\,, \quad \mathscr{J}_{\det'} = I_{n-1}(S)\,, \quad (5.6)$$

and, by Laplace expansion, π and π' map to the given target. Since both $\det$ and $\det'$ are homogeneous, they admit standard Euler vector fields

$$\chi = \sum_{i,j} x_{i,j} \frac{\partial}{\partial x_{i,j}}\,, \quad \chi' = \sum_{i \leq j} x_{i,j} \frac{\partial}{\partial x_{i,j}}\,.$$

Note that

$$\pi((\delta_{i,j}), 0) = \chi\,, \quad \pi'((\delta_{i,j})) = 2\chi\,. \quad (5.7)$$

By Gulliksen–Negård [40] and Józefiak [41] respectively, there are exact sequences

$$\mathrm{Sl}_n(\mathscr{O}_X)^{\oplus 2} \longrightarrow \mathrm{Mat}_n(\mathscr{O}_X) \longrightarrow I_{n-1}(M) \longrightarrow 0\,,$$
$$(A,B) \longmapsto MA - BM\,,$$
$$C \longmapsto \mathrm{tr}(M^\star C)\,,$$
$$\mathrm{Sl}_n(\mathscr{O}_{X'}) \longrightarrow \mathrm{Sym}_n(\mathscr{O}_{X'}) \longrightarrow I_{n-1}(S) \longrightarrow 0\,,$$
$$A \longmapsto SA + A^t S\,,$$
$$D \longmapsto \mathrm{tr}(S^\star D)\,,$$

where, using eq. (5.5),

$$\mathrm{tr}(M^\star C) = \sum_{i,j} m^\star_{j,i} c_{i,j} = \sum_{i,j} c_{i,j} \frac{\partial \det}{\partial x_{i,j}}\,,$$
$$\mathrm{tr}(S^\star D) = \sum_{i,j} s^\star_{i,j} d_{i,j} = \sum_{i\leq j}(2 - \delta_{i,j})s^\star_{i,j} d_{i,j} \quad (5.8)$$
$$= \sum_{i\leq j} d_{i,j} \frac{\partial \det'}{\partial x_{i,j}}\,.$$

Using eqs. (5.4) and (5.6) this means that

$$\pi(\mathrm{Sl}_n(\mathscr{O}_X)^{\oplus 2}) = \mathrm{Ann}_{\Theta_X}(\det)\,,$$
$$\pi'(\mathrm{Sl}_n(\mathscr{O}_{X'})) = \mathrm{Ann}_{\Theta_{X'}}(\det')\,.$$

With eqs. (5.3) and (5.7) surjectivity of π and π' follows. $\qquad\square$

Dropping the genericity hypothesis, we now consider polynomial matrix families

$$Y = \mathbb{K}^m \xrightarrow{\;M\;} \mathrm{Mat}_n(\mathbb{K}) = X\,,$$
$$Y = \mathbb{K}^m \xrightarrow{\;S\;} \mathrm{Sym}_n(\mathbb{K}) = X'\,.$$

They are defined by matrices $M \in \mathrm{Mat}_n(\mathscr{O})$ and $S \in \mathrm{Sym}_n(\mathscr{O})$ with entries in $\mathscr{O} = \mathscr{O}_Y$ (cf. eq. (5.1)). By abuse of notation, we set

$$\mathscr{J}_M = \left\langle \frac{\partial M}{\partial y_k} \;\middle|\; k = 1, \ldots, m \right\rangle \subset \mathrm{Mat}_n(\mathscr{O})\,,$$
$$\mathscr{J}_S = \left\langle \frac{\partial S}{\partial y_k} \;\middle|\; k = 1, \ldots, m \right\rangle \subset \mathrm{Sym}_n(\mathscr{O})\,.$$

Consider the (truncated) Gulliksen–Negård and Józefiak complexes

$$\mathrm{Sl}_n(\mathscr{O})^{\oplus 2} \xrightarrow{\;\rho\;} \mathrm{Mat}_n(\mathscr{O}) \longrightarrow I_{n-1}(M) \longrightarrow 0\,, \quad (5.9)$$
$$\mathrm{Sl}_n(\mathscr{O}) \xrightarrow{\;\rho'\;} \mathrm{Sym}_n(\mathscr{O}) \longrightarrow I_{n-1}(S) \longrightarrow 0\,.$$

Proposition 2.
(a) If $I_{n-1}(M)$ has (the maximal) codimension 4, then there is a surjective map

$$\rho^{-1}(\mathscr{J}_M) \xrightarrow{\qquad \pi \qquad} \!\!\!\!\!\rightarrow \mathrm{Ann}_\Theta(\det M)\,,$$
$$(A,B) \mapsto MA - BM = \sum_{k=1}^m c_k \frac{\partial M}{\partial y_k} \mapsto \sum_{k=1}^m c_k \frac{\partial}{\partial y_k}\,.$$

In particular, if $\det M$ admits an Euler vector field $\chi \in \Theta$, then $\mathrm{Der}(-\log(\det M))$ is generated by χ and the image of π.

(b) If $I_{n-1}(S)$ has (the maximal) codimension 3, then there is a surjective map

$$\rho'^{-1}(\mathscr{J}_S) \xrightarrow{\qquad \pi' \qquad} \!\!\!\!\!\rightarrow \mathrm{Ann}_\Theta(\det S)\,,$$
$$A \longmapsto SA + A^t S = \sum_{k=1}^m c_k \frac{\partial S}{\partial y_k} \mapsto \sum_{k=1}^m c_k \frac{\partial}{\partial y_k}\,.$$

In particular, if $\det S$ admits an Euler vector field $\chi \in \Theta$, then $\mathrm{Der}(-\log(\det S))$ is generated by χ and the image of π'.

Proof. Using eq. (5.8), the chain rule yields

$$\mathrm{tr}\left(M^\star \frac{\partial M}{\partial y_k}\right) = \sum_{i,j} \frac{\partial m_{i,j}}{\partial y_k} \frac{\partial \det}{\partial x_{i,j}}(M) = \frac{\partial \det M}{\partial y_k}\,,$$
$$\mathrm{tr}\left(S^\star \frac{\partial S}{\partial y_k}\right) = \sum_{i\leq j} \frac{\partial s_{i,j}}{\partial y_k} \frac{\partial \det}{\partial x_{i,j}}(S) = \frac{\partial \det S}{\partial y_k}\,. \quad (5.10)$$

By Gulliksen–Negård [40] and Józefiak [41] respectively, the hypotheses imply that the complexes (5.9) are exact. By eq. (5.10) they induce exact sequences

$$\rho^{-1}(\mathscr{J}_M) \xrightarrow{\;\rho\;} \mathscr{J}_M \longrightarrow \mathscr{J}_{\det M} \longrightarrow 0\,,$$
$$\rho'^{-1}(\mathscr{J}_S) \xrightarrow{\;\rho'\;} \mathscr{J}_S \longrightarrow \mathscr{J}_{\det S} \longrightarrow 0\,.$$

With eq. (5.4) surjectivity of π and π' follows. The particular claims are due to eq. (5.3). $\qquad\square$

Finally we specialize to the case of interest in our context.

Corollary 3. *Assume that $S \in \mathrm{Sym}_n(\mathscr{O})$ has a block form*

$$S = (s_{i,j}) = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix}\,,$$

where $S_{1,1}$ is constant invertible and $s_{i,j} = x_{i,j} = y_{\sigma_{i,j}}$ for $i \leq j$ with (i,j) in block column 2. Then $\mathrm{Der}(-\log(\det S))$ is generated by all

$$\pi'(A) = \sum_{k=1}^m c_k \frac{\partial}{\partial y_k}\,,$$

where (cf. Proposition 2.(b))

$$\sum_{k=1}^m c_k \frac{\partial S}{\partial y_k} = SA + A^t S\,, \quad A = \begin{pmatrix} 0 & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix} \in \mathrm{Mat}_n(\mathscr{O})\,. \quad (5.11)$$

Proof. By Micali–Villamayor (see Lemma (1.1) of ref. [41]), there is an invertible matrix C such that

$$C^t S C = \begin{pmatrix} S_0 & 0 \\ 0 & S' \end{pmatrix}\,.$$

The matrix S_0 is still constant invertible and $S' \equiv S_{2,2}$ modulo the variables $x_{i,j}$ with (i,j) in block $(1,2)$. The entries of $S' \in \mathrm{Sym}_{n'}(\mathcal{O})$ are thus algebraically independent over the polynomial ring over $\mathbb{K}$ in these variables. By Józefiak (Thm. (2.3) of ref. [41]) it follows that $I_{n-1}(S) = I_{n'-1}(S')$ has codimension 3.

For well-definedness of π', it suffices to verify that $\pi'(A) \in \mathrm{Der}(-\log(\det S))$ if $A = (\delta_{i,k}\delta_{j,\ell})$. In this case

$$SA + A^t S = \sum_{i=1}^{n}(1 + \delta_{i,\ell})y_{\sigma_{i,k}}\frac{\partial S}{\partial y_{\sigma_{i,\ell}}},$$

and hence, using eq. (5.5) and Laplace expansion,

$$\pi'(A)(\det S) = \sum_{i=1}^{n}(1 + \delta_{i,\ell})y_{\sigma_{i,k}}\frac{\partial \det S}{\partial y_{\sigma_{i,\ell}}}$$
$$= \sum_{i=1}^{n}(1 + \delta_{i,\ell})(2 - \delta_{i,\ell})s_{i,k}s_{i,\ell}^{\star} = 2\delta_{k,\ell}\det S.$$

Note that $\det S$ admits the Euler vector field

$$\chi = \pi'((\delta_{i,n}\delta_{j,n})) = \sum_{i=1}^{n}(1 + \delta_{i,n})y_{\sigma_{i,n}}\frac{\partial}{\partial y_{\sigma_{i,n}}}.$$

So the hypotheses of Proposition 2.(b) are satisfied.

The module $\mathscr{J}_S$ consists of all symmetric matrices with $(1,1)$-block 0. Writing $A \in \mathrm{Sl}_n(\mathcal{O})$ in block form

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix},$$

eq. (5.11) reduces to

$$S_{1,1}A_{1,1} + S_{1,2}A_{2,1} + A_{1,1}^t S_{1,1} + A_{2,1}^t S_{2,1} = 0. \quad (5.12)$$

For any $W \in \mathrm{Skw}_n(\mathcal{O})$, adding WS to A leaves $SA + A^t S$ invariant. Using

$$W = \begin{pmatrix} -A_{1,1}S_{1,1}^{-1} - S_{1,1}^{-1}A_{2,1}^t S_{2,1}S_{1,1}^{-1} & S_{1,1}^{-1}A_{2,1}^t \\ -A_{2,1}S_{1,1}^{-1} & 0 \end{pmatrix}$$

makes $A_{*,1} = 0$ and turns eq. (5.12) into $0 = 0$. $\qquad\square$

Returning to the setup of Sec. II, consider the matrix S in eq. (2.3) with the given block form. Its submatrix $S_{1,1}$ is the Gram matrix G in eq. (2.7) whose entries are the Mandelstam variables $\lambda_{i,j}$ in eq. (2.6) which are treated as constants in the integration and IBP reduction. As noted below eq. (2.8), $U = \det G$ is nonvanishing provided that $p_1, \ldots, p_E$ are linearly independent. Let $\mathbb{K} = \mathbb{Q}(\lambda_{i,j})$ be the field of rational functions in the Mandelstam variables over the rational numbers. Note that the characteristic of $\mathbb{K}$ is 0, so that the above assumption on the characteristic is satisfied.[2] Then $S_{1,1}$ is constant invertible and Corollary 3 applies. As a result the $L(L+E)$ syzygies in eqs. (4.6)–(4.7) generate all syzygies in eq. (3.6).

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[2] Note that while in the actual integration the variables corre-

VI. EXAMPLES

In this section we work out explicit expressions for the syzygy generators presented in Sec. IV for three diagrams.

A. Fully massless planar double box

As a simple example we consider the fully massless planar double-box diagram shown in fig. 1.
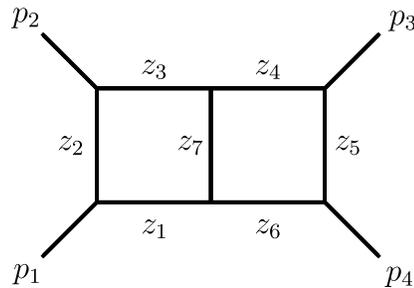

Figure 1: The fully massless planar double-box diagram. All external momenta are taken to be outgoing.

In this case, the combined number of propagators and irreducible scalar products (2.10) is $m = 9$. In agreement with eq. (2.12), we define the z-variables as follows, setting $P_{1,2} \equiv p_1 + p_2$,

$$\begin{aligned} &z_1 = \ell_1^2, &z_2 = (\ell_1 - p_1)^2, \quad &z_3 = (\ell_1 - P_{1,2})^2, \\ &z_4 = (\ell_2 + P_{1,2})^2, &z_5 = (\ell_2 - p_4)^2, \quad &z_6 = \ell_2^2, \\ &z_7 = (\ell_1 + \ell_2)^2, &z_8 = (\ell_1 + p_4)^2, \quad &z_9 = (\ell_2 + p_1)^2. \end{aligned}$$
$$(6.1)$$

We choose as the set (2.2) of all independent external and loop momenta $V = (p_1, p_2, p_3, \ell_1, \ell_2)$. The lexicographically-ordered set of elements (i, j) in eq. (2.9) then becomes,

$$(x_1, \ldots, x_9) \equiv (x_{1,4}, \ x_{1,5}, \ x_{2,4}, \ x_{2,5}, \ x_{3,4}, \ x_{3,5}, \\ x_{4,4}, \ x_{4,5}, \ x_{5,5}), \qquad (6.2)$$

and it immediately follows that the matrices in eq. (2.11)

___

sponding to the external momenta may take real values and the remaining ones may take complex values, the IBP relations have a generating system defined over the rationals. Both the Laplace expansion and a syzygy module computation via Gröbner basis methods lead to such a generating system. This again simplifies the computation of solutions satisfying further constraints.

are given by,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ -2 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (6.3)$$

and, for $\alpha = 1, \ldots, 9$,

$$B_\alpha = 0 \quad \text{for} \quad \alpha \notin \{3,4\} \quad \text{and} \quad B_3 = B_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.4}$$

and $m_\alpha = 0$. We can now use eqs. (2.11)–(2.12) to express the syzygy generators in eqs. (4.6)–(4.7) in terms of the z_α, yielding,

$$t_{4,1} = (z_1 - z_2, \, z_1 - z_2, \, -s + z_1 - z_2, \, 0, \, 0, \, 0,$$
$$z_1 - z_2 - z_6 + z_9, \, t + z_1 - z_2, \, 0, \, 0),$$

$$t_{4,2} = (s + z_2 - z_3, \, z_2 - z_3, \, z_2 - z_3, \, 0, \, 0, \, 0,$$
$$z_2 - z_3 + z_4 - z_9, \, -t + z_2 - z_3, \, 0, \, 0),$$

$$t_{4,3} = (-s + z_3 - z_8, \, t + z_3 - z_8, \, z_3 - z_8, \, 0, \, 0, \, 0,$$
$$z_3 - z_4 + z_5 - z_8, \, z_3 - z_8, \, 0, \, 0),$$

$$t_{4,4} = (2z_1, \, z_1 + z_2, \, -s + z_1 + z_3, \, 0, \, 0, \, 0,$$
$$z_1 - z_6 + z_7, \, z_1 + z_8, \, 0, \, -2),$$

$$t_{4,5} = (-z_1 - z_6 + z_7, \, -z_1 + z_7 - z_9, \, s - z_1 - z_4 + z_7,$$
$$0, \, 0, \, 0, \, -z_1 + z_6 + z_7, \, -z_1 - z_5 + z_7, \, 0, \, 0), \quad (6.5)$$

$$t_{5,1} = (0, \, 0, \, 0, \, s - z_6 + z_9, \, -t - z_6 + z_9, \, z_9 - z_6,$$
$$z_1 - z_2 - z_6 + z_9, \, 0, \, z_9 - z_6, \, 0),$$

$$t_{5,2} = (0, \, 0, \, 0, \, z_4 - z_9, \, t + z_4 - z_9, \, -s + z_4 - z_9,$$
$$z_2 - z_3 + z_4 - z_9, \, 0, \, z_4 - z_9, \, 0),$$

$$t_{5,3} = (0, \, 0, \, 0, \, z_5 - z_4, \, z_5 - z_4, \, s - z_4 + z_5,$$
$$z_3 - z_4 + z_5 - z_8, \, 0, \, -t - z_4 + z_5, \, 0),$$

$$t_{5,4} = (0, \, 0, \, 0, \, s - z_3 - z_6 + z_7, \, -z_6 + z_7 - z_8, \, -z_1 - z_6 + z_7,$$
$$z_1 - z_6 + z_7, \, 0, \, -z_2 - z_6 + z_7, \, 0),$$

$$t_{5,5} = (0, \, 0, \, 0, \, -s + z_4 + z_6, \, z_5 + z_6, \, 2z_6,$$
$$-z_1 + z_6 + z_7, \, 0, \, z_6 + z_9, \, -2).$$

Syzygies obtained from S-polynomial-based computations are not guaranteed to be of degree one. For example, from the SINGULAR command `syz` one can obtain a representation with 13 generators of up to cubic degree. More specifically, `syz` produces 10 generators of degree one, two generators of degree two, and one generator of degree three.

Expressions for on-shell syzygies are too lengthy to record here, but we give a few examples: on the cut $\mathcal{S}_{\text{cut}} = \{1, 4, 7\}$ one can find a representation of $\widehat{\mathcal{T}} \cap \mathcal{Z}$ with 18 generators of up to cubic degree, and on the cut $\mathcal{S}_{\text{cut}} = \{2, 5, 7\}$ a representation of $\widehat{\mathcal{T}} \cap \mathcal{Z}$ with 20 generators of up to cubic degree.

B. Planar double box with internal mass

As a more non-trivial example we consider a planar double-box diagram with propagators of equal mass as shown in fig. 2.
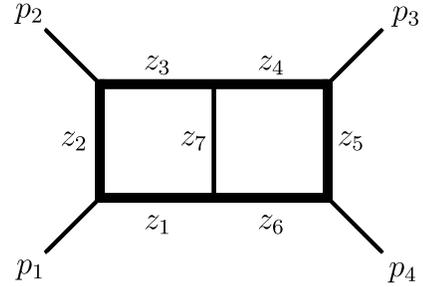
Figure 2: Planar double-box diagram. The bold lines represent massive propagators with the same mass M. All external momenta are taken to be outgoing.

As in the massless case in Sec. VI A, the combined number of propagators and irreducible scalar products (2.10) is $m = 9$. In analogy with eq. (2.12), we define the z-variables as follows, setting $P_{1,2} \equiv p_1 + p_2$,

$$\begin{aligned} z_1 &= \ell_1^2 - M^2, & z_2 &= (\ell_1 - p_1)^2 - M^2, \\ z_3 &= (\ell_1 - P_{1,2})^2 - M^2, & z_4 &= (\ell_2 + P_{1,2})^2 - M^2, \\ z_5 &= (\ell_2 - p_4)^2 - M^2, & z_6 &= \ell_2^2 - M^2, \\ z_7 &= (\ell_1 + \ell_2)^2, & z_8 &= (\ell_1 + p_4)^2, \\ z_9 &= (\ell_2 + p_1)^2. \end{aligned}$$
$$(6.6)$$

Again we choose as the set (2.2) of all independent external and loop momenta $V = (p_1, p_2, p_3, \ell_1, \ell_2)$. The lexicographically-ordered set of elements $x_{i,j}$ in eq. (2.9) is again that in eq. (6.2), and the matrices in eq. (2.11) are given by eqs. (6.3) and (6.4), whereas m_α in eq. (2.11) is given by $m_\alpha = M$ for $1 \le \alpha \le 6$ and $m_\beta = 0$ for $7 \le \beta \le 9$.

Using eqs. (2.11)–(2.12) to express the syzygy generators in eqs. (4.6)–(4.7) in terms of the z_α, we find in the

case at hand,

$$t_{4,1} = (z_1 - z_2,\ z_1 - z_2,\ -s + z_1 - z_2,\ 0,\ 0,\ 0,$$
$$z_1 - z_2 - z_6 + z_9 - M^2,\ t + z_1 - z_2,\ 0,\ 0),$$

$$t_{4,2} = (s + z_2 - z_3,\ z_2 - z_3,\ z_2 - z_3,\ 0,\ 0,\ 0,$$
$$z_2 - z_3 + z_4 - z_9 + M^2,\ -t + z_2 - z_3,\ 0,\ 0),$$

$$t_{4,3} = (-s + z_3 - z_8 + M^2,\ t + z_3 - z_8 + M^2,\ z_3 - z_8 + M^2,$$
$$0,\ 0,\ 0,\ z_3 - z_4 + z_5 - z_8 + M^2,\ z_3 - z_8 + M^2,\ 0,\ 0),$$

$$t_{4,4} = (2z_1 + 2M^2,\ z_1 + z_2 + 2M^2,\ -s + z_1 + z_3 + 2M^2,$$
$$0,\ 0,\ 0,\ z_1 - z_6 + z_7,\ z_1 + z_8 + M^2,\ 0,\ -2),$$

$$t_{4,5} = (-z_1 - z_6 + z_7 - 2M^2,\ -z_1 + z_7 - z_9 - M^2,$$
$$s - z_1 - z_4 + z_7 - 2M^2,\ 0,\ 0,\ 0,\ -z_1 + z_6 + z_7,$$
$$-z_1 - z_5 + z_7 - 2M^2,\ 0,\ 0),\qquad (6.7)$$

$$t_{5,1} = (0,\ 0,\ 0,\ s - z_6 + z_9 - M^2,\ -t - z_6 + z_9 - M^2,$$
$$z_9 - z_6 - M^2,\ z_1 - z_2 - z_6 + z_9 - M^2,\ 0,\ z_9 - z_6 - M^2,\ 0),$$

$$t_{5,2} = (0,\ 0,\ 0,\ z_4 - z_9 + M^2,\ t + z_4 - z_9 + M^2,$$
$$-s + z_4 - z_9 + M^2,\ z_2 - z_3 + z_4 - z_9 + M^2,\ 0,$$
$$z_4 - z_9 + M^2,\ 0),$$

$$t_{5,3} = (0,\ 0,\ 0,\ z_5 - z_4,\ z_5 - z_4,\ s - z_4 + z_5,$$
$$z_3 - z_4 + z_5 - z_8 + M^2,\ 0,\ -t - z_4 + z_5,\ 0),$$

$$t_{5,4} = (0,\ 0,\ 0,\ s - z_3 - z_6 + z_7 - 2M^2,\ -z_6 + z_7 - z_8 - M^2,$$
$$z_7 - z_1 - z_6 - 2M^2,\ z_1 - z_6 + z_7,\ 0,\ z_7 - z_2 - z_6 - 2M^2,\ 0),$$

$$t_{5,5} = (0,\ 0,\ 0,\ -s + z_4 + z_6 + 2M^2,\ z_5 + z_6 + 2M^2,$$
$$2z_6 + 2M^2,\ -z_1 + z_6 + z_7,\ 0,\ z_6 + z_9 + M^2,\ -2),$$

which agrees with eq. (6.5) in the case $M = 0$.

C. Fully massless non-planar double pentagon

As a yet more non-trivial example we consider the fully massless non-planar double-pentagon diagram shown in fig. 3.

In this case, the combined number of propagators and irreducible scalar products (2.10) is $m = 11$. In agreement with eq. (2.12), we define the z-variables as follows, setting $P_{i,j} \equiv p_i + p_j$,

$$z_1 = \ell_1^2, \qquad z_2 = (\ell_1 - p_1)^2, \qquad z_3 = (\ell_1 - P_{1,2})^2,$$
$$z_4 = (\ell_2 - P_{3,4})^2, \quad z_5 = (\ell_2 - p_4)^2, \qquad z_6 = \ell_2^2,$$
$$z_7 = (\ell_1 + \ell_2)^2, \quad z_8 = (\ell_1 + \ell_2 + p_5)^2, \quad z_9 = (\ell_1 + p_3)^2,$$
$$z_{10} = (\ell_1 + p_4)^2, \quad z_{11} = (\ell_2 + p_1)^2.$$
$$(6.8)$$

We choose as the set (2.2) of all independent external and loop momenta $V = (p_1, p_2, p_3, p_4, \ell_1, \ell_2)$. The lexicographically-ordered set of elements (i, j) in eq. (2.9)
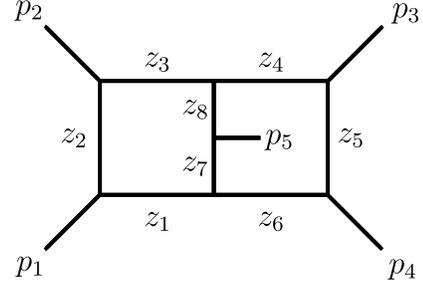

Figure 3: The fully massless non-planar double-pentagon diagram. All external momenta are taken to be outgoing.

then becomes,

$$(x_1, \ldots, x_{11}) \equiv (x_{1,5},\ x_{1,6},\ x_{2,5},\ x_{2,6},\ x_{3,5},\ x_{3,6},$$
$$x_{4,5},\ x_{4,6},\ x_{5,5},\ x_{5,6},\ x_{6,6}),\quad (6.9)$$

and it immediately follows that the matrices in eq. (2.11) are given by,

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$
$$(6.10)$$

and, for $\alpha = 1, \ldots, 11$,

$$B_\alpha = 0 \quad \text{for } \alpha \notin \{3, 4\} \quad \text{and} \qquad (6.11)$$

$$B_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (6.12)$$

and $m_\alpha = 0$. We can now use eqs. (2.11)–(2.12) to express the syzygy generators in eqs. (4.6)–(4.7) in terms

of the z_α, yielding,

$$t_{5,1} = (z_1 - z_2, \; z_1 - z_2, \; -s_{1,2} + z_1 - z_2, \; 0, \; 0, \; 0,$$
$$z_1 - z_2 - z_6 + z_{11}, \; -s_{1,2} - s_{1,3} - s_{1,4} + z_1 - z_2 - z_6 + z_{11},$$
$$s_{1,3} + z_1 - z_2, \; s_{1,4} + z_1 - z_2, \; 0, \; 0),$$

$$t_{5,2} = (s_{1,2} + z_2 - z_3, \; z_2 - z_3, \; z_2 - z_3, \; 0, \; 0, \; 0,$$
$$-s_{3,4} + z_1 + z_2 + z_4 + z_7 - z_8 - z_9 - z_{10} - z_{11},$$
$$s_{1,3} + s_{1,4} + z_1 + z_2 + z_4 + z_7 - z_8 - z_9 - z_{10} - z_{11},$$
$$s_{1,2} + s_{2,3} + z_2 - z_3, \; -s_{1,3} - s_{1,4} - s_{2,3} - s_{3,4} + z_2 - z_3, \; 0, \; 0),$$

$$t_{5,3} = (z_9 - z_1, \; -s_{1,3} - z_1 + z_9, \; -s_{1,3} - s_{2,3} - z_1 + z_9, \; 0, \; 0, \; 0,$$
$$s_{3,4} - z_1 - z_4 + z_5 + z_9, \; -s_{1,3} - s_{2,3} - z_1 - z_4 + z_5 + z_9,$$
$$z_9 - z_1, \; s_{3,4} - z_1 + z_9, \; 0, \; 0),$$

$$t_{5,4} = (z_{10} - z_1, \; -s_{1,4} - z_1 + z_{10}, \; -s_{1,4} - s_{2,4} - z_1 + z_{10},$$
$$0, \; 0, \; 0, \; -z_1 - z_5 + z_6 + z_{10},$$
$$s_{1,2} + s_{1,3} + s_{2,3} - z_1 - z_5 + z_6 + z_{10},$$
$$s_{3,4} - z_1 + z_{10}, \; z_{10} - z_1, \; 0, \; 0),$$

$$t_{5,5} = (2z_1, \; z_1 + z_2, \; -s_{1,2} + z_1 + z_3, \; 0, \; 0, \; 0,$$
$$z_1 - z_6 + z_7, \; -s_{1,2} + 2z_1 + z_3 - z_6 + z_7 - z_9 - z_{10},$$
$$z_1 + z_9, \; z_1 + z_{10}, \; 0, \; -2),$$

$$t_{5,6} = (-z_1 - z_6 + z_7, \; -z_1 + z_7 - z_{11},$$
$$s_{1,2} + s_{3,4} - 2z_1 - z_3 - z_4 + z_8 + z_9 + z_{10}, \; 0, \; 0, \; 0,$$
$$-z_1 + z_6 + z_7, \; s_{1,2} - 2z_1 - z_3 + z_6 + z_8 + z_9 + z_{10},$$
$$s_{3,4} - z_1 - z_4 + z_5 - z_6 + z_7, \; -z_1 - z_5 + z_7, \; 0, \; 0), \qquad (6.13)$$

$$t_{6,1} = (0, \; 0, \; 0, \; -s_{1,3} - s_{1,4} - z_6 + z_{11}, \; -s_{1,4} - z_6 + z_{11},$$
$$z_{11} - z_6, \; z_1 - z_2 - z_6 + z_{11},$$
$$-s_{1,2} - s_{1,3} - s_{1,4} + z_1 - z_2 - z_6 + z_{11}, \; 0, \; 0, \; z_{11} - z_6, \; 0),$$

$$t_{6,2} = (0, 0, 0, \; s_{1,3} + s_{1,4} + z_1 + z_3 + z_4 + z_7 - z_8 - z_9 - z_{10} - z_{11},$$
$$s_{1,3} + s_{1,4} + s_{2,3} + z_1 + z_3 + z_4 + z_7 - z_8 - z_9 - z_{10} - z_{11},$$
$$-s_{1,2} - s_{3,4} + z_1 + z_3 + z_4 + z_7 - z_8 - z_9 - z_{10} - z_{11},$$
$$-s_{3,4} + z_1 + z_2 + z_4 + z_7 - z_8 - z_9 - z_{10} - z_{11},$$
$$s_{1,3} + s_{1,4} + z_1 + z_2 + z_4 + z_7 - z_8 - z_9 - z_{10} - z_{11}, \; 0, \; 0,$$
$$-s_{3,4} + z_1 + z_3 + z_4 + z_7 - z_8 - z_9 - z_{10} - z_{11}, \; 0),$$

$$t_{6,3} = (0, \; 0, \; 0, \; z_5 - z_4, \; z_5 - z_4, \; s_{3,4} - z_4 + z_5,$$
$$s_{3,4} - z_1 - z_4 + z_5 + z_9, \; -s_{1,3} - s_{2,3} - z_1 - z_4 + z_5 + z_9, \; 0, \; 0,$$
$$s_{1,3} + s_{3,4} - z_4 + z_5, \; 0),$$

$$t_{6,4} = (0, \; 0, \; 0, \; -s_{3,4} - z_5 + z_6, \; z_6 - z_5, \; z_6 - z_5,$$
$$-z_1 - z_5 + z_6 + z_{10}, \; s_{1,2} + s_{1,3} + s_{2,3} - z_1 - z_5 + z_6 + z_{10}, \; 0, \; 0,$$
$$s_{1,4} - z_5 + z_6, \; 0),$$

$$t_{6,5} = (0, \; 0, \; 0, \; z_1 - z_6 + z_7 - z_9 - z_{10}, \; -z_6 + z_7 - z_{10},$$
$$-z_1 - z_6 + z_7, \; z_1 - z_6 + z_7, \; -s_{1,2} + 2z_1 + z_3 - z_6 + z_7 - z_9 - z_{10},$$
$$0, \; 0, \; -z_2 - z_6 + z_7, \; 0),$$

$$t_{6,6} = (0, \; 0, \; 0, \; -s_{3,4} + z_4 + z_6, \; z_5 + z_6, \; 2z_6, \; -z_1 + z_6 + z_7,$$
$$s_{1,2} - 2z_1 - z_3 + z_6 + z_8 + z_9 + z_{10}, \; 0, \; 0, \; z_6 + z_{11}, \; -2).$$

In contrast, for this example the SINGULAR command `syz` did not produce output after 56 hours of running time with 52 GB of RAM used.

VII. CONCLUSIONS

Integration-by-parts (IBP) identities between loop integrals arise from the vanishing integration of total derivatives in dimensional regularization. The condition that a total derivative leads to an IBP identity which does not involve dimension shifts can be stated as the syzygy equation (3.6). We presented in eqs. (4.6)–(4.7) an explicit generating set of solutions of the syzygy equation, valid for any number of loops and external momenta. In general, S-polynomial computations would be required in order to obtain the syzygies. However, as the Baikov polynomial (2.5) is the determinant of a matrix, a generating set of syzygies can be obtained from the Laplace expansion of the determinant. Moreover, we showed that the syzygies needed for IBP identities evaluated on a generalized-unitarity cut can be obtained immediately from eqs. (4.6)–(4.7) by a straightforward module intersection computation.

We emphasize that the closed-form expressions in eqs. (4.6)–(4.7) are valid for any number of loops and external legs. The only quantities that depend on the graph in question are the relations of the z-variables to the x-variables in eqs. (2.11)–(2.12). In particular, the closed-form expressions allow the construction of purely D-dimensional IBP identities in cases where S-polynomial based computations of syzygies are not feasible. An example of the latter is the non-planar double-pentagon diagram considered in Sec. VI C.

Moreover, an important feature of the syzygies eqs. (4.6)–(4.7) is that they are guaranteed to have degree at most one. In contrast, a generating set of syzygies obtained from an S-polynomial computation would in general have higher degrees. The fact that the syzygies obtained here are of degree at most one is useful because as a result the computation of solutions which satisfy further constraints is dramatically simplified. For example, one may be interested in imposing the further constraint on the total derivatives that no integrals with squared propagators are encountered in the IBP identities.

It is worth pointing out that ref. [42] makes use of syzygies to construct IBP identities which involve arbitrary numerator powers. These are then solved as difference equations to obtain the IBP reductions. Based on preliminary tests of several two-loop examples, the syzygies from Laplacian expansion considered here can also produce recursive relations similar to the relations in ref. [42]. This direction merits further investigation.

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